Numerical Solution of Laplace Equation

The electric potential in a charge-free region is given by the Laplace equation

$$\nabla^2 V = 0 \tag{1}$$

In Cartesian coordinates in two dimensions, this equation expands as

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{2}$$

An analytical solution of this partial differential equation can be found only if the domain of interest (i.e., the region in which the solution is to be computed) is simple, such as a rectangle or a circle. For finding the voltage distribution in a region of nonuniform geometry, engineers frequently use numerical techniques.

There are several numerical methods that can be used for solving equation (1) among which the finite difference method method is the simplest. The following is a brief introduction to the finite difference method.

The finite difference method consists of three steps:

- Discretize the domain as a grid.
- Find an algebraic equation for each node in the grid.
- Solve the set of algebraic equations.

1 Grid

For finite difference solution, we direct our attention only at some finite number of points in the domain. These points are referred to as grid (or mesh or node) points. For domains in Cartesian geometry, a rectangular grid is usually used. For cylindrical geometry, the region is discretized utilizing cylindrical coordinates.

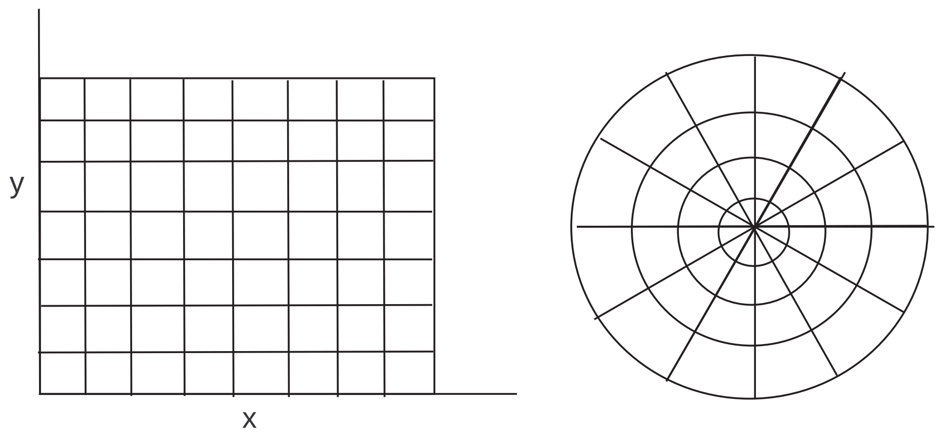

Fig 1. Discretization

It is not necessary that the grid size be uniform for the x and y coordinates, however a uniform grid results in some simplification in the numerical method. Note however that for domains with irregular shaped geometry, a uniform grid cannot be used. Nevertheless, it is possible to extend the method with good accuracy.

The potential V at the boundary nodes are given. The objective is to find the potential V for the interior nodes.

2 Difference Equation

This step generates an algebraic equation for each node in the domain. This is based on the Taylor series expansion:

$$V(x+\Delta x) = V(x) + \frac{\partial V}{\partial x}(x)\Delta x + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(x)(\Delta x)^2 + \text{ higher order terms}$$

$$V(x-\Delta x) = V(x) - \frac{\partial V}{\partial x}(x)\Delta x + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(x)(\Delta x)^2 + \text{ higher order terms} \tag{3}$$

Adding the two equations and neglecting higher order terms, we obtain

$$\frac{\partial^2 V}{\partial x^2}(x) = \frac{V(x+\Delta x) - 2V(x) + V(x-\Delta x)}{(\Delta x)^2} \tag{4}$$

Note that V is also a function of y so that the above equation actually holds for any given y. Thus we rewrite the above equation as

$$\frac{\partial^2 V}{\partial x^2}(x,y) = \frac{V(x+\Delta x,y) - 2V(x,y) + V(x-\Delta x,y)}{(\Delta x)^2} \tag{5}$$

Similar expansion can be done for the potential in the y-coordinate to obtain

$$\frac{\partial^2 V}{\partial y^2}(x,y) = \frac{V(x,y+\Delta y) - 2V(x,y) + V(x,y-\Delta y)}{(\Delta y)^2} \tag{6}$$

Substituting the above two equations in (2), we obtain

$$\frac{V(x+\Delta x,y) - 2V(x,y) + V(x-\Delta x,y)}{(\Delta x)^2} + \frac{V(x,y+\Delta y) - 2V(x,y) + V(x,y-\Delta y)}{(\Delta y)^2} = 0 \tag{7}$$

If we choose $\Delta x = \Delta y$, the above equation simplifies to

$$V(x+\Delta x,y) + V(x-\Delta x,y) + V(x,y+\Delta y) + V(x,y-\Delta y) - 4V(x,y) = 0 \tag{8}$$

Denote the grid points in x coordinate as $i = 1, 2, \cdots, N_x$ and those for the y-coordinate as $j = 1, 2, \cdots, N_y$. Then the above equation is expressed as

$$V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j} = 0 \tag{9}$$

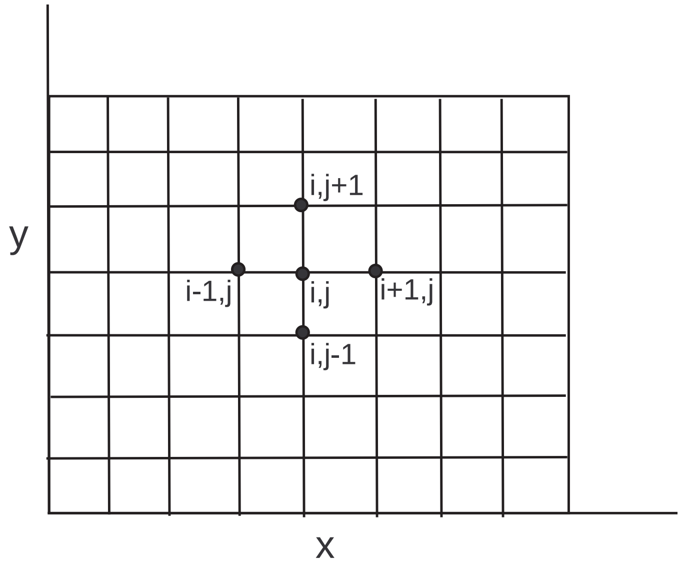

Fig 2. Difference Formula

This equation is called the difference equation that approximates the Laplace equation at the grid point (i, j). Clearly it shows that the potential V at any grid point is the average of that of four neighboring nodes.

For the numerical solution of the Laplace equation, a difference equation must be derived for each interior node of the domain which are then solved simultaneously for the unknown potential V.

Note also that in deriving the difference formula (9), we have neglected all higher order terms in the Taylor series expansion. As such, the numerical solution obtained by this method is only an approximate solution. Nevertheless, accuracy of the computed solution can be improved by discretizing the domain to a finer grid, i.e. by taking Δx and Δy small. This also means that there will be many more equations that must be solved simultaneously which increases computer time and demands more memory.

3 Solution

The next step is to solve the difference equations derived in the previous step. Again, there are various ways these equations can be solved. For example, one may write the difference equations as a linear matrix equation and find the solution by inverting the matrix equation. An easier alternative is to find the solution using an iterative method.

For the iterative solution, first assume some values for the unknown potential V for the interior nodes; in fact $V_{i,j} = 0$ is also a possible initial guess of the solution. Then sequentially update the solution for each node by taking the average of four nearest neighboring nodes:

$$V_{i,j} = \frac{1}{4}\left[V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1}\right] \tag{10}$$

Assume a rectangular domain with potential on its four boundaries are known. This means that potential at all boundary nodes are known, and we have to calculate the potential at interior nodes only. Based on the above analysis, here is the pseudo-code for the algorithm:

1. Assume initial guess of the solution
2. Update the solution using

   for i=2 to $N_x - 1$
   for j=2 to $N_y - 1$
   $V(i,j) = \frac{1}{4}\left[V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1}\right]$
   end for
   end for

3. Repeat from Step 2 until the solution converges

For convergence test, take the sum of square of potential of all nodes, i.e., $\sum_i \sum_j V_{i,j}^2$ for each iteration, and compare with that computed in the previous iteration. Stop the iteration when increment of the squared sum of potentials for two successive iterations is small (say 10^{-5}).

To calculate the electric field intensity, we use the equation

$$\mathbf{E} = -\nabla V \tag{11}$$

For the two dimensional geometry, this equation simplifies to

$$\mathbf{E} = -\frac{\partial V}{\partial x}\mathbf{a}_x - \frac{\partial V}{\partial y}\mathbf{a}_y \tag{12}$$

Using equations (3), we then obtain

$$\mathbf{E}(x,y) = -\frac{V(x+\Delta x, y) - V(x-\Delta x, y)}{2\Delta x}\mathbf{a}_x - \frac{V(x, y+\Delta y) - V(x, y-\Delta y)}{2\Delta y}\mathbf{a}_y \tag{13}$$

A pseudo-code for computer implementation of this calculation is as follows

for i=2 to $N_x - 1$
  for j=2 to $N_y - 1$
    $E(i,j) = [-\frac{V_{i+1,j} - V_{i-1,j}}{2\Delta x}, -\frac{V_{i,j+1} - V_{i,j-1}}{2\Delta y}]$
  end for
end for

Note that **E** is a vector of two elements, one for the each axis of the rectangular coordinate system. The **E**-field can also be expressed in terms of its magnitude and the associated angle.

Accuracy of numerical solution depends on the grid size Δx and Δy. A coarse grid (i.e. large Δx and Δy) will lead to fast computation of the numerical solution, however will not be accurate. On the other hand a finer grid will require long time for computation but can result in a more accurate solution. If analytical solution is not known, one can solve the problem for using successively and compare solutions. An acceptable solution is obtained when the solution does not change significantly if grid size is reduced any further.

You can test accuracy of your Matlab code for the following problem that can be solved analytically: Write a Matlab code to find the electric potential $V(x,y)$ and the electric field $E(x,y)$ at all grid points in the domain. Compare your numerical solution and the analytical solution (given below). Consider the rectangular domain as shown with the applied voltage on the boundary:

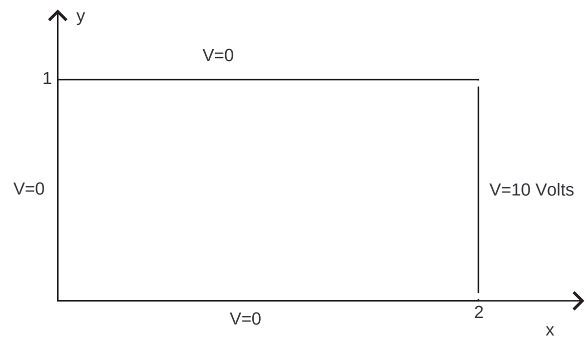

Fig 3. Domain

Analytical solution: It can be shown that the analytical solution of this problem is given by

$$V(x,y) = \sum_{m=1,3,5,\cdots}^{\infty} \frac{4V_0}{m\pi} \cdot \frac{1}{\sinh \frac{m\pi a}{b}} \cdot \sinh \frac{m\pi x}{b} \cdot \sin \frac{m\pi y}{b} \tag{14}$$

where $a = 2$ is the width and $b = 1$ is the height of the region, and $V_0 = 10$ is the applied voltage.

4 Project

Consider the rectangular domain as shown with the applied voltage on the boundary:

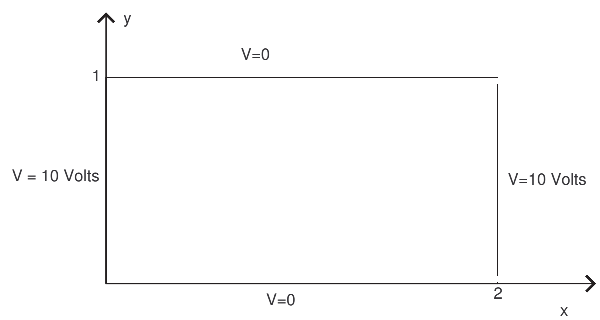

Fig 4. Domain

- Write a Matlab code to find the electric potential $V(x, y)$ and the electric field $E(x, y)$ at all grid points in the domain.
- Draw equipotential lines in the domain.
- Draw contour plot of constant field intensity.
- How much energy is stored in this device?
- Could this device be considered as a capacitor? If so, find its capacitance.
- Disucss anything else about this problem that you find interesting or important.