3 Kinematics

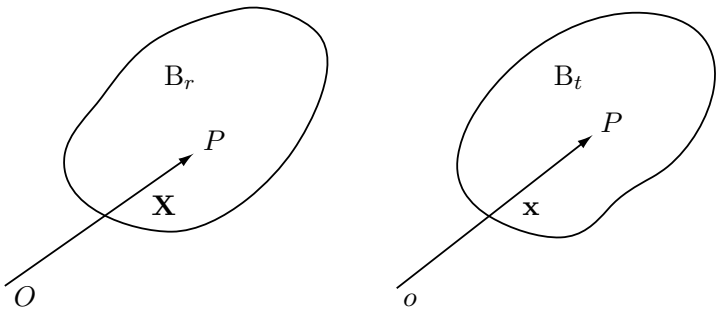
3.1 Bodies, configurations and motions

Definition: A *body* $\mathcal{B}$ is a set whose elements can be put into one-to-one correspondence with points of a region B in three-dimensional Euclidean point space.

The elements of $\mathcal{B}$ are called *particles* (or *material points*), and B is called a *configuration* of $\mathcal{B}$.

As the body moves the configuration changes with time. Let $t \in I \subset \mathbb{R}$ denote time, where I is an interval in $\mathbb{R}$. If, with each $t \in I$, we associate a unique configuration B_t of $\mathcal{B}$ then the family of configurations $\{\mathrm{B}_t : t \in I\}$ is called a *motion* of $\mathcal{B}$. We assume that as $\mathcal{B}$ moves continuously then B_t changes continuously.

It is convenient to identify a *reference configuration*, B_r say, which is an arbitrarily chosen fixed configuration. Then, any particle P of $\mathcal{B}$ may be labelled by its position vector $\mathbf{X}$ in B_r relative to some origin O. Let $\mathbf{x}$ be the position vector of P in the configuration B_t at time t relative to an origin o (which need not coincide with O).



We say that $\mathcal{B}$ *occupies* the configuration B_t at time t — B_t is also referred to as the *current configuration*.

[Note that B_r need not be a configuration actually occupied by $\mathcal{B}$ during the motion, but is often chosen to be the configuration occupied by $\mathcal{B}$ at some prescribed time.]

Since B_r and B_t are configurations of $\mathcal{B}$ there exists a bijection mapping $\boldsymbol{\chi} : \mathrm{B}_r \to \mathrm{B}_t$ such that

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\chi}(\mathbf{X}) \qquad \text{for all } \mathbf{X} \in \mathrm{B}_r, \\ \mathbf{X} &= \boldsymbol{\chi}^{-1}(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \mathrm{B}_t. \end{aligned} \tag{3.1}$$

The mapping $\boldsymbol{\chi}$ is called the *deformation* of the body *from* B_r *to* B_t.

Since B_t depends on t we write

$$\mathbf{x} = \boldsymbol{\chi}_t(\mathbf{X}), \quad \mathbf{X} = \boldsymbol{\chi}_t^{-1}(\mathbf{x}) \tag{3.2}$$

instead of (3.1), or

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \quad \text{for all } \mathbf{X} \in \mathrm{B}_r,\ t \in I. \tag{3.3}$$

For each particle P (with label $\mathbf{X}$) this describes the motion of P with t as parameter, and hence the motion of $\mathcal{B}$. It is usual to assume that $\boldsymbol{\chi}(\mathbf{X}, t)$ is twice-continuously differentiable with respect to position and time.

Example: *Rigid motion*

A motion is said to be it rigid if the distance between any two particles of $\mathcal{B}$ is invariant.

The motion defined by

$$\mathbf{x} \equiv \boldsymbol{\chi}(\mathbf{X}, t) = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{X}, \tag{3.4}$$

where $\mathbf{c}(t)$ is a vector and $\mathbf{Q}(t)$ is a proper orthogonal CT(2), is a rigid motion.

To show this we consider $\mathbf{Y} \in \mathrm{B}_r$ so that

$$\mathbf{y} = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{Y}.$$

Then

$$\begin{aligned}
|\mathbf{x} - \mathbf{y}|^2 &= (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\
&= [\mathbf{Q}(\mathbf{X} - \mathbf{Y})] \cdot [\mathbf{Q}(\mathbf{X} - \mathbf{Y})] \\
&= [\mathbf{Q}^T\mathbf{Q}(\mathbf{X} - \mathbf{Y})] \cdot (\mathbf{X} - \mathbf{Y}) \quad \text{(using (1.17))} \\
&= (\mathbf{X} - \mathbf{Y}) \cdot (\mathbf{X} - \mathbf{Y}) \quad (\text{since } \mathbf{Q}^T\mathbf{Q} = \mathbf{I}) \\
&= |\mathbf{X} - \mathbf{Y}|^2.
\end{aligned}$$

In fact, although we have not proved it, *every* rigid motion can be expressed in the form (3.4). Note that $\mathbf{c}(t)$ represents a *translation* and $\mathbf{Q}(t)$ a *rotation*.

In the development of the basic principles of continuum mechanics a body $\mathcal{B}$ is endowed with various physical properties which are represented by scalar, vector and tensor fields defined on *either* B_r *or* B_t (for example, density, temperature, shape of surface). In the case of B_r the position vector $\mathbf{X}$ and time t serve as independent variables, and the fields are then said to be defined in terms of the *referential* or *material* description. Alternatively, in the case of B_t, $\mathbf{x}$ and t are used and the description is said to be *spatial*. The terminologies *Lagrangian* and *Eulerian descriptions* are also used in respect of B_r and B_t respectively.

Rectangular Cartesian coordinate systems with basis vectors $\{\mathbf{E}_i\}$ and $\{\mathbf{e}_i\}$ are chosen for B_r and B_t respectively, with *material coordinates* X_i and *spatial coordinates* x_i. Thus, relative to origins O and o respectively, we have

$$\mathbf{X} = X_i\mathbf{E}_i, \quad \mathbf{x} = x_i\mathbf{e}_i. \tag{3.5}$$

[Note that other vectors may be referred to either basis and tensors may be referred to either or to both simultaneously. Thus, a vector field $\mathbf{u}$ and a CT(2) $\mathbf{T}$ may be written

$$\mathbf{u} = u_i\mathbf{E}_i = \hat{u}_i\mathbf{e}_i,$$

$$\mathbf{T} = T_{ij}\mathbf{E}_i \otimes \mathbf{E}_j = \tilde{T}_{ij}\mathbf{E}_i \otimes \mathbf{e}_j = \bar{T}_{ij}\mathbf{e}_i \otimes \mathbf{E}_j = \hat{T}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j,$$

for example.]

3.2 The material derivative

The *velocity* $\mathbf{v}$ of a particle P is defined as

$$\mathbf{v} \equiv \dot{\mathbf{x}} = \frac{\partial}{\partial t}\boldsymbol{\chi}(\mathbf{X}, t), \tag{3.6}$$

i.e. the rate of change of position of P (or $\partial/\partial t$ at fixed $\mathbf{X}$). The *acceleration* $\mathbf{a}$ of P is

$$\mathbf{a} \equiv \dot{\mathbf{v}} \equiv \ddot{\mathbf{x}} = \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(\mathbf{X}, t), \tag{3.7}$$

where the dot indicates differentiation with respect to t at fixed $\mathbf{X}$.

Let ϕ be a scalar field defined on B_t, i.e. $\phi(\mathbf{x}, t)$. Since $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, we may write

$$\phi(\mathbf{x}, t) = \phi[\boldsymbol{\chi}(\mathbf{X}, t), t] \equiv \Phi(\mathbf{X}, t), \tag{3.8}$$

which defines the notation Φ. Thus, any field defined on B_t (respectively B_r) can, through (3.2), equally be defined on B_r (respectively B_t).

The *material derivative* of ϕ is the rate of change of ϕ at fixed *material point* P, i.e. at fixed $\mathbf{X}$. We write the material derivative as $\dot{\phi}$ or $D\phi/Dt$.

By definition

$$\dot{\phi} = \frac{\partial}{\partial t}\Phi(\mathbf{X}, t).$$

By the chain rule for partial derivatives we then have

$$\begin{aligned} \tfrac{\partial}{\partial t}\Phi(\mathbf{X}, t) &= \tfrac{\partial}{\partial t}\phi(\mathbf{x}, t) + \tfrac{\partial x_i}{\partial t}\tfrac{\partial}{\partial x_i}\phi(\mathbf{x}, t) \\ &= \tfrac{\partial}{\partial t}\phi(\mathbf{x}, t) + \tfrac{\partial \mathbf{x}}{\partial t} \cdot \nabla\phi(\mathbf{x}, t). \end{aligned}$$

Using (3.6) we thus have

$$\frac{\partial}{\partial t}\Phi(\mathbf{X}, t) \equiv \dot{\phi} \equiv \frac{D\phi}{Dt} = \frac{\partial}{\partial t}\phi + \mathbf{v} \cdot \nabla\phi. \tag{3.9}$$

Similarly, for a vector field

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}[\boldsymbol{\chi}(\mathbf{X}, t), t] = \mathbf{U}(\mathbf{X}, t) \tag{3.10}$$

(which defines $\mathbf{U}$), we obtain

$$\frac{\partial}{\partial t}\mathbf{U}(\mathbf{X}, t) \equiv \dot{\mathbf{u}} \equiv \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{u}. \tag{3.11}$$

In particular, the acceleration $\mathbf{a} = \dot{\mathbf{v}}$ is given by

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}. \tag{3.12}$$

We emphasize that

$$\frac{\partial}{\partial t}\Big|_{\mathbf{X}} = \frac{D}{Dt} \equiv \frac{\partial}{\partial t}\Big|_{\mathbf{x}} + \mathbf{v} \cdot \nabla. \tag{3.13}$$

3.3 Differentiation of Cartesian tensor fields

Let $\phi, \mathbf{u}, \mathbf{T}$ be scalar, vector and tensor functions of position $\mathbf{x}$. The operation of the gradient operator, grad or ∇, on these functions with respect to the basis $\{\mathbf{e}_i\}$ is defined as follows:

$$\operatorname{grad} \phi \equiv \nabla \phi = \frac{\partial \phi}{\partial x_i}\mathbf{e}_i,$$

$$\begin{aligned} \operatorname{grad} \mathbf{u} \equiv \nabla \otimes \mathbf{u} &= \tfrac{\partial \mathbf{u}}{\partial x_q} \otimes \mathbf{e}_q \\ &= \tfrac{\partial}{\partial x_q}(u_p \mathbf{e}_p) \otimes \mathbf{e}_q \\ &= \tfrac{\partial u_p}{\partial x_q}\mathbf{e}_p \otimes \mathbf{e}_q, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \operatorname{grad} \mathbf{T} \equiv \nabla \otimes \mathbf{T} &= \tfrac{\partial}{\partial x_i}\mathbf{T} \otimes \mathbf{e}_i \\ &= \tfrac{\partial}{\partial x_i}(T_{pq}\mathbf{e}_p \otimes \mathbf{e}_q) \otimes \mathbf{e}_i \\ &= \tfrac{\partial T_{pq}}{\partial x_i}\mathbf{e}_p \otimes \mathbf{e}_q \otimes \mathbf{e}_i, \end{aligned} \tag{3.15}$$

and similarly for higher-order tensors. Note that the operation of grad increases the order of the tensor by one. *Contraction* of $\operatorname{grad} \mathbf{u}$ gives $\nabla \cdot \mathbf{u}$. There are several possible contractions of $\nabla \otimes \mathbf{T}$. We *define* $\operatorname{div} \mathbf{T}$ as follows.

$$\operatorname{div} \mathbf{T} \equiv \frac{\partial T_{pq}}{\partial x_i}\mathbf{e}_q(\mathbf{e}_p \cdot \mathbf{e}_i)$$

— the p—i contraction. Since $\mathbf{e}_p \cdot \mathbf{e}_i = \delta_{ip}$ we obtain

$$\operatorname{div} \mathbf{T} = \frac{\partial T_{pq}}{\partial x_p}\mathbf{e}_q. \tag{3.16}$$

Exercise Show that (3.11) can be written as

$$\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} + (\nabla \otimes \mathbf{u})\mathbf{v}. \tag{3.17}$$

3.4 Deformation gradient

Let Grad, Div, Curl (respectively grad, div, curl) denote the gradient, divergence and curl operators in the reference (respectively current) configuration, i.e. with respect to $\mathbf{X}$ (respectively $\mathbf{x}$).

We define the *deformation gradient tensor* $\mathbf{F}$ as

$$\mathbf{F}(\mathbf{X}, t) = \operatorname{Grad} \mathbf{x} \equiv \operatorname{Grad} \boldsymbol{\chi}(\mathbf{X}, t). \tag{3.18}$$

With respect to the chosen basis vectors and with use of (3.14) we have

$$\mathbf{F} = \frac{\partial}{\partial X_j}(x_i \mathbf{e}_i) \otimes \mathbf{E}_j = \frac{\partial x_i}{\partial X_j} \mathbf{e}_i \otimes \mathbf{E}_j$$

or, in component form,

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \tag{3.19}$$

with $x_i = \chi_i(\mathbf{X}, t)$.

We assume that $\det \mathbf{F} \neq 0$ (to be justified shortly) so that $\mathbf{F}$ has an inverse $\mathbf{F}^{-1}$, given by

$$\mathbf{F}^{-1} = \operatorname{grad} \mathbf{X}, \tag{3.20}$$

with components

$$(\mathbf{F}^{-1})_{ij} = \frac{\partial X_i}{\partial x_j}. \tag{3.21}$$

This may be checked by means of the calculation

$$(\mathbf{F}\mathbf{F}^{-1})_{ij} = F_{ik}(\mathbf{F}^{-1})_{kj} = \frac{\partial x_i}{\partial X_k}\frac{\partial X_k}{\partial x_j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

It follows from (3.19) that

$$F_{ij} dX_j = \frac{\partial x_i}{\partial X_j} dX_j = dx_i,$$

i.e.

$$d\mathbf{x} = \mathbf{F} d\mathbf{X}, \tag{3.22}$$

which has inverse

$$d\mathbf{X} = \mathbf{F}^{-1} d\mathbf{x}. \tag{3.23}$$

Equation (3.22) describes how small *line elements* $d\mathbf{X}$ of material at $\mathbf{X}$ transform under the deformation into $d\mathbf{x}$ (which consists of the same material as $d\mathbf{X}$) at $\mathbf{x}$.

Line elements transform *linearly* since $\mathbf{F}$ depends on $\mathbf{X}$ (and not on $d\mathbf{X}$). Thus, at each $\mathbf{X}$, $\mathbf{F}$ is a *linear mapping* (i.e. a second-order tensor).

If $\mathbf{F}$ is independent of $\mathbf{X}$ then the deformation is said to be *homogeneous* (the same at each point of the body).

We justify taking $\mathbf{F}$ to be *non-singular* ($\det \mathbf{F} \neq 0$) by noting that $\mathbf{F}d\mathbf{X} \neq \mathbf{0}$ if $d\mathbf{X} \neq \mathbf{0}$ — a line element cannot be annihilated.

Example

Let $\phi, \mathbf{u}, \mathbf{T}$ be CT(0), CT(1), CT(2) fields associated with a moving body. We establish the following formulas:

$$\begin{aligned}
&\operatorname{Grad}\phi = \mathbf{F}^T \operatorname{grad}\phi, \quad \operatorname{Grad}\mathbf{u} = (\operatorname{grad}\mathbf{u})\mathbf{F}, \\
&\operatorname{Div}\mathbf{u} = J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{u}), \quad \operatorname{Div}\mathbf{T} = J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{T}), \quad (3.24)
\end{aligned}$$

where J is defined as

$$J = \det \mathbf{F}. \tag{3.25}$$

First, we calculate

$$\begin{aligned}
\mathbf{F}^T \operatorname{grad}\phi &= \left(\tfrac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j\right)^T \tfrac{\partial \phi}{\partial x_p}\mathbf{e}_p \\
&= \tfrac{\partial x_i}{\partial X_j}(\mathbf{E}_j \otimes \mathbf{e}_i)\mathbf{e}_p \tfrac{\partial \phi}{\partial x_p} \\
&= \tfrac{\partial x_i}{\partial X_j}\tfrac{\partial \phi}{\partial x_p}\mathbf{E}_j \delta_{ip} \\
&= \tfrac{\partial x_p}{\partial X_j}\tfrac{\partial \phi}{\partial x_p}\mathbf{E}_j = \tfrac{\partial \phi}{\partial X_j}\mathbf{E}_j = \operatorname{Grad}\phi.
\end{aligned}$$

Next,

$$\begin{aligned}
(\operatorname{grad}\mathbf{u})\mathbf{F} &= \left(\tfrac{\partial u_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j\right)\left(\tfrac{\partial x_p}{\partial X_q}\mathbf{e}_p \otimes \mathbf{E}_q\right) \\
&= \tfrac{\partial u_i}{\partial x_j}\tfrac{\partial x_p}{\partial X_q}(\mathbf{e}_i \otimes \mathbf{e}_j)(\mathbf{e}_p \otimes \mathbf{E}_q) \\
&= \tfrac{\partial u_i}{\partial x_j}\tfrac{\partial x_p}{\partial X_q}\mathbf{e}_i \otimes \mathbf{E}_q \delta_{jp} \\
&= \tfrac{\partial u_i}{\partial x_p}\tfrac{\partial x_p}{\partial X_q}\mathbf{e}_i \otimes \mathbf{E}_q \\
&= \tfrac{\partial u_i}{\partial X_q}\mathbf{e}_i \otimes \mathbf{E}_q = \operatorname{Grad}\mathbf{u}.
\end{aligned}$$

For the right-hand side of the third equation in (3.24), we calculate

$$\begin{aligned}
J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{u}) &= J\tfrac{\partial}{\partial x_p}(J^{-1}F_{pq}u_q) \\
&= F_{pq}\tfrac{\partial u_q}{\partial x_p} + Ju_q\tfrac{\partial}{\partial x_p}(J^{-1}F_{pq}). \ (*)
\end{aligned}$$

But,

$$\begin{aligned}
\tfrac{\partial}{\partial x_p}(J^{-1}F_{pq}) &= \tfrac{\partial X_r}{\partial x_p}\tfrac{\partial}{\partial X_r}(J^{-1}F_{pq}) \\
&= -J^{-2}\tfrac{\partial J}{\partial X_r}\tfrac{\partial X_r}{\partial x_p}F_{pq} + J^{-1}\tfrac{\partial X_r}{\partial x_p}\tfrac{\partial F_{pq}}{\partial X_r} \\
&= -J^{-2}\left(J(\mathbf{F}^{-1})_{ts}\tfrac{\partial F_{st}}{\partial X_r}\right)\underbrace{\frac{\partial X_r}{\partial x_p}\frac{\partial x_p}{\partial X_q}}_{\delta_{rq}} + J^{-1}\tfrac{\partial X_r}{\partial x_p}\tfrac{\partial^2 x_p}{\partial X_q \partial X_r}
\end{aligned}$$

In the above, we have used $F_{pq} = \partial x_p/\partial X_q$ and the relation

$$\frac{\partial}{\partial t}(\det \mathbf{T}) = (\det \mathbf{T})\mathrm{tr}\left(\mathbf{T}^{-1}\frac{\partial \mathbf{T}}{\partial t}\right). \tag{3.26}$$

Thus,

$$\frac{\partial}{\partial x_p}(J^{-1}F_{pq}) = -J^{-1}\frac{\partial X_t}{\partial x_s}\frac{\partial^2 x_s}{\partial X_q \partial X_t} + J^{-1}\frac{\partial X_r}{\partial x_p}\frac{\partial^2 x_p}{\partial X_q \partial X_r} = 0.$$

Hence, $(*)$ gives

$$J\mathrm{div}\,(J^{-1}\mathbf{F}\mathbf{u}) = F_{pq}\frac{\partial u_q}{\partial x_p} = \frac{\partial x_p}{\partial X_q}\frac{\partial u_q}{\partial x_p} = \frac{\partial u_q}{\partial X_q} = \mathrm{Div}\,\mathbf{u}.$$

Similarly,

$$\begin{aligned} J\mathrm{div}\,(J^{-1}\mathbf{F}\mathbf{T}) &= J\tfrac{\partial}{\partial x_p}(J^{-1}F_{pq}T_{qr}\mathbf{E}_r) \\ &= \underbrace{J\frac{\partial}{\partial x_p}(J^{-1}F_{pq})T_{qr}\mathbf{E}_r}_{=0} + F_{pq}\tfrac{\partial T_{qr}}{\partial x_p}\mathbf{E}_r \\ &= \tfrac{\partial x_p}{\partial X_q}\tfrac{\partial T_{qr}}{\partial x_p}\mathbf{E}_r = \tfrac{\partial T_{qr}}{\partial X_q}\mathbf{E}_r = \mathrm{Div}\,\mathbf{T}. \end{aligned}$$

Proof of (3.26) For a small increment δ in time t we have

$$\mathbf{F}(\mathbf{X}, \tau + \delta) = \mathbf{F}(\mathbf{X}, \tau) + \delta\dot{\mathbf{F}}(\mathbf{X}, \tau) + \mathcal{O}(\delta^2).$$

Then

$$\frac{\partial J}{\partial t} = \lim_{\delta\to 0}\frac{1}{\delta}\left[J(\mathbf{X}, t+\delta) - J(\mathbf{X}, t)\right] = \lim_{\delta\to 0}\frac{1}{\delta}\left[\det(\mathbf{F} + \delta\dot{\mathbf{F}}) - J\right].$$

But

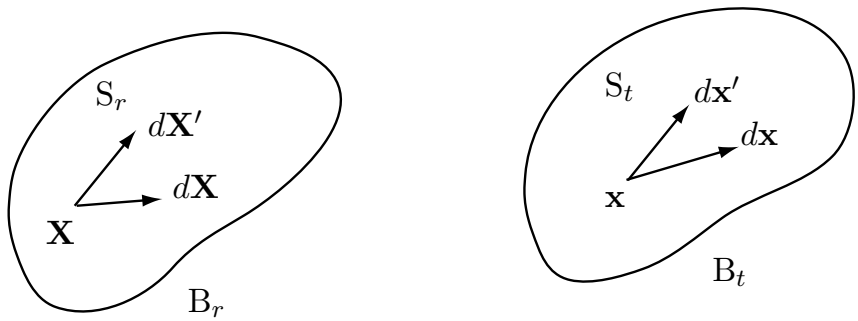
$$\begin{aligned} \det(\mathbf{F} + \delta\dot{\mathbf{F}}) &= \det[\mathbf{F}(\mathbf{I} + \delta\mathbf{F}^{-1}\dot{\mathbf{F}})] = (\det \mathbf{F})\det(\mathbf{I} + \delta\mathbf{F}^{-1}\dot{\mathbf{F}}) \\ &= J\det[\delta(\mathbf{F}^{-1}\dot{\mathbf{F}} + \delta^{-1}\mathbf{I})] = J\delta^3\det(\mathbf{F}^{-1}\dot{\mathbf{F}} + \delta^{-1}\mathbf{I}) \\ &= \delta^3 J[\delta^{-3} + \delta^{-2}I_1(\mathbf{F}^{-1}\dot{\mathbf{F}}) + \delta^{-1}I_2(\mathbf{F}^{-1}\dot{\mathbf{F}}) + I_3(\mathbf{F}^{-1}\dot{\mathbf{F}})] \\ &= J[1 + \delta\, I_1(\mathbf{F}^{-1}\dot{\mathbf{F}}) + \mathcal{O}(\delta^2)], \\ &= J[1 + \delta\,\mathrm{tr}(\mathbf{F}^{-1}\dot{\mathbf{F}}) + \mathcal{O}(\delta^2)]. \end{aligned}$$

Now the limit is easily computed as (3.26).

By following similar arguments one can readily get the components of Grad$J$:

$$\frac{\partial}{\partial X_r}(\det \mathbf{F}) = (\det \mathbf{F})\,\mathrm{tr}\left(\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial X_r}\right).$$

For an alternative proof, see Q8 in Example Sheet 3.



3.5 Deformation of area and volume elements

Consider a surface S_r in B_r which deforms into the surface S_t in B_t. Let $\mathbf{X}$ be a point on S_r and $\mathbf{x}$ the corresponding point on S_t. Let $d\mathbf{X}$ and $d\mathbf{X}'$ be line elements of material on S_r based at $\mathbf{X}$ and let $d\mathbf{x}$ and $d\mathbf{x}'$ be their images (on S_t) under the deformation. If $\mathbf{F}$ denotes the deformation gradient, then

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}, \quad d\mathbf{x}' = \mathbf{F}d\mathbf{X}'. \tag{3.27}$$

Let dA and da be surface area elements on S_r and S_t respectively, and let $\mathbf{N}$ and $\mathbf{n}$ be unit normals at $\mathbf{X}$ and $\mathbf{x}$ respectively.

For the parallelogram with sides $d\mathbf{X}, d\mathbf{X}'$ we have

$$\mathbf{N}dA = d\mathbf{X} \times d\mathbf{X}'.$$

Under the deformation this becomes a parallelogram with sides $d\mathbf{x}, d\mathbf{x}'$ and area

$$\mathbf{n}da = d\mathbf{x} \times d\mathbf{x}'.$$

From (3.27) we obtain

$$\begin{aligned} \mathbf{F}^T\mathbf{n}da &= \mathbf{F}^T[(\mathbf{F}d\mathbf{X}) \times (\mathbf{F}d\mathbf{X}')] \\ &= (\det \mathbf{F})d\mathbf{X} \times d\mathbf{X}' \end{aligned}$$

using the result from question 12 of Problem Sheet 3. Hence

$$\mathbf{n}da = J(\mathbf{F}^T)^{-1}\mathbf{N}dA,$$

where $J = \det \mathbf{F}$. With the notation

$$\mathbf{F}^{-T} = (\mathbf{F}^T)^{-1} = (\mathbf{F}^{-1})^T,$$

this becomes

$$\mathbf{n}da = J\mathbf{F}^{-T}\mathbf{N}dA. \tag{3.28}$$

This is an important result — it is known as *Nanson's formula*, and it describes how elements of surface area deform.

Next, consider the parallelepiped in B_r formed by line elements $d\mathbf{X}, d\mathbf{X}', d\mathbf{X}''$ at $\mathbf{X}$. Its volume dV is given by

$$dV = d\mathbf{X} \cdot (d\mathbf{X}' \times d\mathbf{X}'') = \left| \, d\mathbf{X} \; d\mathbf{X}' \; d\mathbf{X}'' \, \right| .$$

The corresponding volume dv in B_t is

$$dv = d\mathbf{x} \cdot (d\mathbf{x}' \times d\mathbf{x}'') = \left| \, d\mathbf{x} \; d\mathbf{x}' \; d\mathbf{x}'' \, \right| = \left| \, \mathbf{F}d\mathbf{X} \; \mathbf{F}d\mathbf{X}' \; \mathbf{F}d\mathbf{X}'' \, \right| = |\mathbf{F}| \, \left| \, d\mathbf{X} \; d\mathbf{X}' \; d\mathbf{X}'' \, \right| ,$$

i.e.

$$dv = JdV. \tag{3.29}$$

By convention, volume is taken to be positive, so that

$$J \equiv \det \mathbf{F} > 0. \tag{3.30}$$

From (3.29) we see that J is a measure of the change in volume under the deformation. If the deformation is such that there is no change in volume then the deformation is said to be *isochoric*, and then

$$J \equiv \det \mathbf{F} = 1. \tag{3.31}$$

For some materials many deformations are such that (3.31) holds to a good approximation, and (3.31) is adopted as an *idealization*. An (ideal) material for which (3.31) holds for *all* deformations is called an *incompressible material*.

Example

A (time-dependent) deformation is defined by

$$x_1 = \alpha X_1 + \beta X_2, \quad x_2 = \alpha^{-1} X_2, \quad x_3 = X_3$$

(the same basis vectors being chosen for $\mathbf{X}$ and $\mathbf{x}$), where $\alpha \, (\neq 0)$ and β are constants. Show that the deformation is isochoric.

With respect to the given Cartesian coordinates the deformation gradient $\mathbf{F}$ has components

$$(F_{ij}) \equiv \left(\frac{\partial x_i}{\partial X_j} \right) = \begin{bmatrix} \alpha & \beta & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Clearly, $\det \mathbf{F} = 1$, so the deformation is isochoric. It corresponds to stretching by a factor α in the x_1 direction, compressing by a factor α^{-1} in the x_2 direction and then shearing by an amount β parallel to the x_1 direction, as illustrated below by application to a square with sides of unit length.

3.6 Further results from tensor algebra

The square root theorem

If $\mathbf{S}$ is a positive definite, symmetric CT(2) then there exists a unique, positive definite, symmetric CT(2), $\mathbf{U}$ say, such that $\mathbf{U}^2 = \mathbf{S}$.

Proof Since $\mathbf{S}$ is symmetric we may write it in the spectral form

$$\mathbf{S} = \sum_{i=1}^{3} s_i \mathbf{e}'_i \otimes \mathbf{e}'_i,$$

where s_i are the (real) eigenvalues of $\mathbf{S}$ and $\{\mathbf{e}'_i\}$ are the (unit) eigenvectors. Since $\mathbf{S}$ is positive definite, we have $s_i > 0$. Now define $\mathbf{U}$ by

$$\mathbf{U} = \sum_{i=1}^{3} \sqrt{s_i} \mathbf{e}'_i \otimes \mathbf{e}'_i.$$

Then, $\mathbf{U}$ is positive definite and symmetric and $\mathbf{U}^2 = \mathbf{S}$, as required. Uniqueness is obvious – convince yourselves of this.

The polar decomposition theorem

Let $\mathbf{F}$ be a CT(2) such that $\det \mathbf{F} > 0$. Then there exist unique, positive definite, symmetric tensors, $\mathbf{U}$ and $\mathbf{V}$, and a unique proper orthogonal tensor $\mathbf{R}$ such that

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}. \tag{3.32}$$

Proof The tensors $\mathbf{F}\mathbf{F}^T$ and $\mathbf{F}^T\mathbf{F}$ are symmetric and positive definite. Hence, by the square root theorem, there exist unique positive definite symmetric tensors $\mathbf{U}$, $\mathbf{V}$ such that

$$\mathbf{V}^2 = \mathbf{F}\mathbf{F}^T, \quad \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}.$$

Now define $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$. We need to prove that $\mathbf{R}$ is proper orthogonal. First, we calculate

$$\mathbf{R}^T\mathbf{R} = (\mathbf{F}\mathbf{U}^{-1})^T(\mathbf{F}\mathbf{U}^{-1}) = \mathbf{U}^{-1}\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{I},$$

and hence we deduce that $\mathbf{R}$ is orthogonal. Second, we calculate

$$\det \mathbf{R} = \det(\mathbf{F}\mathbf{U}^{-1}) = (\det \mathbf{F})(\det \mathbf{U})^{-1} > 0,$$

and it follows that $\mathbf{R}$ is *proper* orthogonal.

Since $\mathbf{U}$ is unique, $\mathbf{R}$ is unique and hence $\mathbf{F} = \mathbf{R}\mathbf{U}$. Similarly, $\mathbf{F} = \mathbf{V}\mathbf{S}$, where $\mathbf{S}$ is proper orthogonal. Thus,

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{S} = \mathbf{R}\mathbf{U}\mathbf{R}^T\mathbf{R}.$$

By uniqueness it follows that $\mathbf{S} = \mathbf{R}$ and hence (3.32) holds. Note that $\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T$.

Corollary

If $\mathbf{U}$ has eigenvalues λ_i and eigenvectors $\mathbf{u}^{(i)}$, $i \in \{1, 2, 3\}$, then $\lambda_i > 0$ and λ_i are also the eigenvalues of $\mathbf{V}$ with eigenvectors $\mathbf{R}\mathbf{u}^{(i)}$.

Proof $\lambda_i > 0$ follows from symmetry and positive definiteness of $\mathbf{U}$. Also, we have

$$\mathbf{V}(\mathbf{R}\mathbf{u}^{(i)}) = \mathbf{V}\mathbf{R}\mathbf{u}^{(i)} = \mathbf{R}\mathbf{U}\mathbf{u}^{(i)} = \mathbf{R}(\lambda_i \mathbf{u}^{(i)}) = \lambda_i(\mathbf{R}\mathbf{u}^{(i)}),$$

which shows that $\mathbf{R}\mathbf{u}^{(i)}$ are the eigenvectors of $\mathbf{V}$.

3.7 Analysis of deformation

3.7.1 Stretch, extension, shear and strain

Let $\mathbf{M}$ and $\mathbf{m}$ be unit vectors along $d\mathbf{X}$ and $d\mathbf{x}$ respectively, so that $d\mathbf{X} = \mathbf{M}|d\mathbf{X}|, d\mathbf{x} = \mathbf{m}|d\mathbf{x}|$ and (3.22) gives

$$\mathbf{m}|d\mathbf{x}| = \mathbf{F}\mathbf{M}|d\mathbf{X}|.$$

Thus

$$|d\mathbf{x}|^2 = (\mathbf{F}\mathbf{M}) \cdot (\mathbf{F}\mathbf{M})|d\mathbf{X}|^2 = (\mathbf{F}^T\mathbf{F}\mathbf{M}) \cdot \mathbf{M}|d\mathbf{X}|^2 \tag{3.33}$$

and hence

$$\frac{|d\mathbf{x}|}{|d\mathbf{X}|} = |\mathbf{F}\mathbf{M}| = [\mathbf{M} \cdot (\mathbf{F}^T\mathbf{F}\mathbf{M})]^{1/2} \equiv \lambda(\mathbf{M}), \tag{3.34}$$

which defines $\lambda(\mathbf{M})$ — the *stretch in the direction* $\mathbf{M}$ at $\mathbf{X}$.

Note that $0 < \lambda(\mathbf{M}) < \infty$.

Now consider a pair of line elements $d\mathbf{X}_1, d\mathbf{X}_2$ based at $\mathbf{X}$, so that

$$d\mathbf{x}_1 = \mathbf{F}d\mathbf{X}_1, \quad d\mathbf{x}_2 = \mathbf{F}d\mathbf{X}_2$$

and the angle between them is given by

$$\cos\Theta = \mathbf{M}_1 \cdot \mathbf{M}_2, \quad \cos\theta = \frac{\mathbf{M}_1 \cdot (\mathbf{F}^T\mathbf{F}\mathbf{M}_2)}{\lambda(\mathbf{M_1})\lambda(\mathbf{M_2})}$$

before and after deformation respectively.

The decrease in angle $\Theta - \theta$ (which may be positive or negative) is called the *shear* of the direction $\mathbf{M}_1, \mathbf{M}_2$ in the plane of $\mathbf{M}_1, \mathbf{M}_2$.

Next, from (3.33) we have

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = d\mathbf{X} \cdot (\mathbf{F}^T\mathbf{F} - \mathbf{I})d\mathbf{X}. \tag{3.35}$$

The material is said to be *unstrained* at $\mathbf{X}$ if no line element changes length, i.e.

$$d\mathbf{X} \cdot (\mathbf{F}^T\mathbf{F} - \mathbf{I})d\mathbf{X} = 0 \quad \text{for all } d\mathbf{X},$$

or, equivalently,

$$\lambda(\mathbf{M}) = 1 \quad \text{for all } \textit{unit} \text{ vectors } \mathbf{M}.$$

It follows that $\mathbf{F}^T\mathbf{F} - \mathbf{I} = \mathbf{O}$, which allows the possibility that $\mathbf{F}$ is just a rotation $\mathbf{R}$, since, for orthogonal $\mathbf{R}$, we have $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

Strain is measured locally by changes in the lengths of line elements, i.e. by the value of (3.35). Thus, the tensor $\mathbf{F}^T\mathbf{F} - \mathbf{I}$ is a measure of strain. The so-called *Green strain tensor* $\mathbf{E}$ is defined by

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}). \tag{3.36}$$

Using the polar decomposition (3.32) for the deformation gradient $\mathbf{F}$, we may also form the following tensor measures of deformation:

$$\begin{aligned} \mathbf{C} &= \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \\ \mathbf{B} &= \mathbf{F}\mathbf{F}^T = \mathbf{V}^2. \end{aligned} \tag{3.37}$$

We refer to $\mathbf{C}$ and $\mathbf{B}$ as the *right* and *left Cauchy-Green deformation tensors* respectively.

Since $\mathbf{U}$ is positive definite and symmetric there exist (unit) eigenvectors $\mathbf{u}^{(i)}$ such that

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{3.38}$$

where $\lambda_i > 0$ are the *principal stretches* of the deformation and $\mathbf{u}^{(i)}$ are the *principal directions*. Note that, in accordance with the definition (3.34), $\lambda_i = \lambda(\mathbf{u}^{(i)})$ — hence the terminology *principal stretch*.

$\mathbf{U}$ and $\mathbf{V}$ are called the *right* and *left stretch tensors* respectively. The deformation $\mathbf{F}$ rotates the principal axes of $\mathbf{U}$ into those of $\mathbf{V}$ as well as stretching along those directions. The principal axes of $\mathbf{U}$ and $\mathbf{V}$ are sometimes referred to as the *Lagrangian* and *Eulerian principal axes* respectively.

The *displacement* $\mathbf{u}$ of a particle is defined as

$$\mathbf{u} = \mathbf{x} - \mathbf{X},$$

so that

$$\mathbf{x} = \mathbf{X} + \mathbf{u}$$

and

$$\mathbf{F} = \operatorname{Grad}\mathbf{x} = \mathbf{I} + \operatorname{Grad}\mathbf{u}, \tag{3.39}$$

where $\operatorname{Grad}\mathbf{u}$ is the *displacement gradient* and $\mathbf{I}$ is the identity tensor.

[Note that $(\operatorname{Grad}\mathbf{X})_{ij} = \partial X_i/\partial X_j = \delta_{ij}$.]

3.7.2 Examples of deformations

1. Homogeneous deformation.

The most general form of homogeneous deformation is given by $\mathbf{x} = \mathbf{F}\mathbf{X} + \mathbf{c}$, with $\mathbf{F}$ independent of $\mathbf{X}$. The following examples are all special cases of this.

2. Homogeneous simple elongation of a circular cylinder (with lateral contraction).

$$\mathbf{F} = \mathbf{U} = \lambda_1 \mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda_2(\mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + \mathbf{u}^{(3)} \otimes \mathbf{u}^{(3)}),$$

with $\mathbf{u}^{(1)}$ along the axis of the cylinder.

3. Homogeneous pure dilatation.

This is defined by $\lambda_1 = \lambda_2 = \lambda_3$, $\mathbf{F} = \lambda_1 \mathbf{I}$ and might be associated with, for example, the deformation of a cube into a cube of a different size.

4. Homogeneous pure shear.

This is defined by $\lambda_1 = \lambda, \lambda_2 = \lambda^{-1}, \lambda_3 = 1$, for example, or

$$\mathbf{F} = \lambda \mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda^{-1} \mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + \mathbf{u}^{(3)} \otimes \mathbf{u}^{(3)}.$$

5. Homogeneous simple shear. Simple shear is defined by the equations

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3,$$

where γ (constant) is called the *amount of shear*, $\tan^{-1}\gamma$ is the shear of the directions $\mathbf{e}_1, \mathbf{e}_2$, and the same basis vectors are used for both reference and current coordinates.

The deformation gradient $\mathbf{F}$ has components

$$(F_{ij}) = \left(\frac{\partial x_i}{\partial X_j}\right) = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is effectively a problem confined to the $(1, 2)$-plane so we now restrict attention to this plane and write

$$(F_{ij}) = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}.$$

To find the Lagrangian principal axes we consider

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & \gamma \\ \gamma & 1+\gamma^2 \end{bmatrix}. \qquad (*)$$

The characteristic equation for $\mathbf{U}^2$, from which the eigenvalues λ^2 are determined, is

$$\det(\mathbf{U}^2 - \lambda^2 \mathbf{I}) = 0,$$

i.e.

$$\begin{vmatrix} 1-\lambda^2 & \gamma \\ \gamma & 1-\lambda^2+\gamma^2 \end{vmatrix},$$

or, when expanded out,

$$\lambda^4 - (2+\gamma^2)\lambda^2 + 1 = 0.$$

Let the roots be λ_1^2, λ_2^2. Then

$$\lambda_1^2 + \lambda_2^2 = 2 + \gamma^2, \quad \lambda_1^2\lambda_2^2 = 1.$$

[Note that $\lambda_3 = 1$ corresponds to $\mathbf{u}^{(3)} = \mathbf{e}_3$.]

Now set $\lambda_1 = \lambda \geq 1$, $\lambda_2 = \lambda^{-1}$ so that

$$\lambda^2 + \lambda^{-2} = 2 + \gamma^2$$

and hence

$$\gamma = \lambda - \lambda^{-1}, \quad \lambda = \frac{1}{2}\gamma + \sqrt{1 + \frac{1}{4}\gamma^2}.$$

Let

$$\mathbf{u}^{(1)} = (\cos\theta, \sin\theta, 0), \quad \mathbf{u}^{(2)} = (-\sin\theta, \cos\theta, 0).$$

Then the representation

$$\mathbf{U}^2 = \lambda_1^2 \mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda_2^2 \mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + \lambda_3^2 \mathbf{u}^{(3)} \otimes \mathbf{u}^{(3)}$$

yields, when restricted to two dimensions,

$$\mathbf{U}^2 = \lambda^2 \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} + \lambda^{-2} \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix}.$$

Comparison with $(*)$ shows that

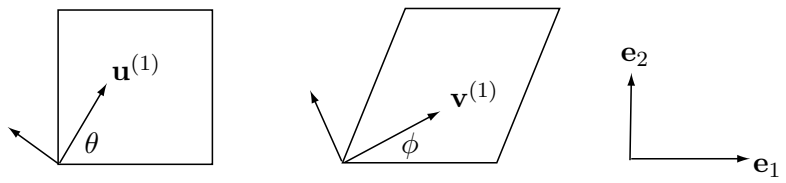
$$\begin{aligned} \lambda^2\cos^2\theta + \lambda^{-2}\sin^2\theta &= 1, \\ \lambda^2\sin^2\theta + \lambda^{-2}\cos^2\theta &= 1 + \gamma^2, \\ (\lambda^2 - \lambda^{-2})\sin\theta\cos\theta &= \gamma, \end{aligned}$$

from which we may deduce that

$$\tan 2\theta = -\frac{2}{\gamma} \quad (\frac{\pi}{4} \leq \theta < \frac{\pi}{2}).$$

The corresponding angle for the principal axes of $\mathbf{F}\mathbf{F}^T = \mathbf{V}^2$ is calculated in a similar way. Let $\mathbf{v}^{(1)} = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2$, $\mathbf{v}^{(2)} = -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2$. The result is

$$\tan 2\phi = \frac{2}{\gamma} \quad (0 < \phi \leq \frac{\pi}{4}).$$



3.8 Analysis of motion

The *velocity gradient tensor*, denoted $\mathbf{L}$, is defined as

$$\mathbf{L} = \operatorname{grad} \mathbf{v} \tag{3.40}$$

and has components

$$L_{ij} = \frac{\partial v_i}{\partial x_j} \tag{3.41}$$

with respect to the basis $\{\mathbf{e}_i\}$.

Using the second equation in (3.24) we obtain

$$\operatorname{Grad} \mathbf{v} = (\operatorname{grad} \mathbf{v})\mathbf{F} = \mathbf{L}\mathbf{F}.$$

Since $\mathbf{v} = \dot{\mathbf{x}}$ we also have

$$\operatorname{Grad} \dot{\mathbf{x}} = \frac{\partial}{\partial t} \operatorname{Grad} \mathbf{x} = \dot{\mathbf{F}}.$$

Hence, we have the important connection

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}. \tag{3.42}$$

Using (3.26)

$$\frac{\partial}{\partial t}(\det \mathbf{F}) = (\det \mathbf{F}) \operatorname{tr}(\mathbf{F}^{-1}\dot{\mathbf{F}})$$

together with (3.42) we deduce that

$$\frac{\partial}{\partial t}(\det \mathbf{F}) = (\det \mathbf{F}) \operatorname{tr}(\mathbf{L})$$

or, equivalently,

$$\dot{J} = J \operatorname{tr}(\mathbf{L}) = J \operatorname{div} \mathbf{v}, \tag{3.43}$$

remembering that $J = \det \mathbf{F}$, $\operatorname{tr}(\mathbf{L}) = L_{ii} = \partial v_i / \partial x_i = \operatorname{div} \mathbf{v}$.

Thus, $\operatorname{div} \mathbf{v}$ measures the rate at which volume changes during the motion.

For an *isochoric* motion $J \equiv 1$, $\dot{J} = 0$ and hence

$$\operatorname{div} \mathbf{v} = 0. \tag{3.44}$$

3.8.1 Stretching and spin

The deformation gradient $\mathbf{F}$ describes how material line elements change their length and orientation during deformation; the velocity gradient $\mathbf{L}$ describes the rate of these changes. Note that while $\mathbf{F}$ relates B_t to B_r, $\mathbf{L}$ is independent of B_r.

Write

$$\mathbf{L} = \mathbf{D} + \mathbf{W}, \tag{3.46}$$

where

$$\mathbf{D} = \underbrace{\frac{1}{2}(\mathbf{L} + \mathbf{L}^T)}_{\text{symmetric}}, \quad \mathbf{W} = \underbrace{\frac{1}{2}(\mathbf{L} - \mathbf{L}^T)}_{\text{skewsymmetric}}. \tag{3.47}$$

In order to interpret $\mathbf{D}$ and $\mathbf{W}$ we consider the line element $d\mathbf{X} \to d\mathbf{x} = \mathbf{F}d\mathbf{X}$. Form the difference

$$\begin{aligned} d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X} &= (\mathbf{F}d\mathbf{X}) \cdot (\mathbf{F}d\mathbf{X}) - d\mathbf{X} \cdot d\mathbf{X} \\ &= d\mathbf{X} \cdot (\mathbf{F}^T\mathbf{F}d\mathbf{X}) - d\mathbf{X} \cdot d\mathbf{X} \\ &= d\mathbf{X} \cdot (\mathbf{F}^T\mathbf{F} - \mathbf{I})d\mathbf{X}. \end{aligned}$$

From (3.42) it follows that

$$\begin{aligned} \tfrac{\partial}{\partial t}(d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X}) &= d\mathbf{X} \cdot \tfrac{\partial}{\partial t}(\mathbf{F}^T\mathbf{F} - \mathbf{I})d\mathbf{X} = d\mathbf{X} \cdot (\mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F})d\mathbf{X} \\ &= d\mathbf{X} \cdot (\mathbf{F}^T\mathbf{L}\mathbf{F} + \mathbf{F}^T\mathbf{L}^T\mathbf{F})d\mathbf{X} = (\mathbf{F}d\mathbf{X}) \cdot (\mathbf{L} + \mathbf{L}^T)\mathbf{F}d\mathbf{X} \\ &= 2d\mathbf{x} \cdot (\mathbf{D}d\mathbf{x}). \end{aligned}$$

This shows that $\mathbf{D}$ measures the rate at which line elements are changing their lengths. It is called the (*Eulerian*) *strain-rate tensor* or *rate of stretching tensor*. The motion is *rigid* if and only if $\mathbf{D} = \mathbf{O}$.

Since

$$\frac{\partial}{\partial t}d\mathbf{x} = \dot{\mathbf{F}}d\mathbf{X} = \mathbf{L}\mathbf{F}d\mathbf{X} = \mathbf{L}d\mathbf{x} = (\mathbf{D} + \mathbf{W})d\mathbf{x}$$

and we have an interpretation of $\mathbf{D}$, as discussed above, it remains to interpret $\mathbf{W}$. We do this by setting $\mathbf{D} = \mathbf{O}$, so that

$$\frac{\partial}{\partial t}d\mathbf{x} = \mathbf{W}d\mathbf{x} = \mathbf{w} \times d\mathbf{x},$$

where $\mathbf{w}$ is the axial vector of $\mathbf{W}$. This shows that the motion is locally a rigid rotation and $\mathbf{W}$ is a measure of the *rate of rotation* (or *spin*) of line elements — $\mathbf{W}$ is called the *body spin*. The combination of $\mathbf{D}$ and $\mathbf{W}$ shows that the motion consists of stretching and rotation (analogous to the interpretation of $\mathbf{U}$ and $\mathbf{R}$).

3.9 Integration of tensors

We first summarize some results from vector calculus which will be needed. The *divergence theorem* is written

$$\int_R \operatorname{div} \mathbf{v} dV = \int_{\partial R} \mathbf{v} \cdot \mathbf{n} dA, \tag{3.48}$$

where R is a domain in $\mathbb{R}^3$ and ∂R is its boundary (a closed surface), and $\mathbf{v}$ is a vector field. An alternative form of the theorem is

$$\int_R \nabla\phi dV = \int_{\partial R} \phi \mathbf{n} dA, \tag{3.49}$$

where ϕ is a scalar field, or, in index notation,

$$\int_R \frac{\partial \phi}{\partial x_i} dV = \int_{\partial R} \phi n_i dA. \tag{3.50}$$

In particular, (3.50) applies to the *components* (which are scalar fields) $T_{pqr\ldots}$ of any CT. Thus,

$$\int_R \frac{\partial T_{pqr\ldots}}{\partial x_i} dV = \int_{\partial R} T_{pqr\ldots} n_i dA. \tag{3.51}$$

In tensor notation (3.51) is equivalent to

$$\int_R \nabla \otimes \mathbf{T} dV = \int_{\partial R} \mathbf{T} \otimes \mathbf{n} dA. \tag{3.52}$$

If, in particular, $\mathbf{T}$ is a CT(2) then contraction of (3.51), putting $i = p$, gives

$$\int_R \frac{\partial T_{pq}}{\partial x_p} dV = \int_{\partial R} T_{pq} n_p dA \tag{3.53}$$

or, in tensor notation,

$$\int_R \operatorname{div} \mathbf{T} dV = \int_{\partial R} \mathbf{T}^T \mathbf{n} dA. \tag{3.54}$$

This is an *important formula* and will occur frequently in the remaining sections of the notes.

For completeness, we also recall Stokes' theorem

$$\int_C \mathbf{u} \cdot d\mathbf{x} = \int_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dA, \tag{3.55}$$

or its equivalent

$$\int_C \phi d\mathbf{x} = \int_S d\mathbf{A} \times \nabla\phi, \tag{3.56}$$

where S is an open surface bounded by the contour C, $\mathbf{u}$ is a vector field and $d\mathbf{A} = \mathbf{n} dA$.

3.10 Transport formulae

Let C_t, S_t and R_t denote curves, surfaces and regions in B_t, the current configuration of the body. Then, the following identities hold:

$$\frac{d}{dt} \int_{C_t} \phi d\mathbf{x} = \int_{C_t} (\dot{\phi} d\mathbf{x} + \phi \mathbf{L} d\mathbf{x}), \tag{3.57}$$

$$\frac{d}{dt}\int_{S_t}\phi\mathbf{n}da = \int_{S_t}\{[\dot{\phi}+\phi\mathrm{tr}\,(\mathbf{L})]\mathbf{n}-\phi\mathbf{L}^T\mathbf{n}\}da, \tag{3.58}$$

$$\frac{d}{dt}\int_{R_t}\phi dv = \int_{R_t}[\dot{\phi}+\phi\mathrm{tr}\,(\mathbf{L})]dv, \tag{3.59}$$

$$\frac{d}{dt}\int_{C_t}\mathbf{u}\cdot d\mathbf{x} = \int_{C_t}(\dot{\mathbf{u}}+\mathbf{L}^T\mathbf{u})\cdot d\mathbf{x}, \tag{3.60}$$

$$\frac{d}{dt}\int_{S_t}\mathbf{u}\cdot\mathbf{n}da = \int_{S_t}[\dot{\mathbf{u}}+\mathbf{u}\mathrm{tr}\,(\mathbf{L})-\mathbf{L}\mathbf{u}]\cdot\mathbf{n}da, \tag{3.61}$$

$$\frac{d}{dt}\int_{R_t}\mathbf{u}dv = \int_{R_t}[\dot{\mathbf{u}}+\mathrm{tr}\,(\mathbf{L})\mathbf{u}]dv. \tag{3.62}$$

Proof Use the formulae $d\mathbf{x}=\mathbf{F}d\mathbf{X}, \mathbf{n}da = J\mathbf{F}^{-T}\mathbf{N}dA, dv = JdV$ to convert the integrals over C_t, S_t, R_t in B_t to integrals over C_r, S_r, R_r in B_r, together with expressions for $\dot{\mathbf{F}}$ and $\dot{J}$. We illustrate the process by proving (3.58).

$$\begin{aligned}
\tfrac{d}{dt}\textstyle\int_{S_t}\mathbf{u}\cdot\mathbf{n}da &= \tfrac{d}{dt}\textstyle\int_{S_r}\mathbf{u}\cdot(J\mathbf{F}^{-T}\mathbf{N})dA \\
&\qquad \text{(note the integral is now over } S_r\text{)} \\
&= \tfrac{d}{dt}\textstyle\int_{S_r}(J\mathbf{F}^{-1}\mathbf{u})\cdot\mathbf{N}dA \\
&\qquad \text{(using the definition of transpose)} \\
&= \textstyle\int_{S_r}\underbrace{\frac{\partial}{\partial t}(J\mathbf{F}^{-1}\mathbf{u})}_{\text{at fixed }\mathbf{X}}\cdot\mathbf{N}dA \\
&= \textstyle\int_{S_r}[J\mathbf{F}^{-1}\dot{\mathbf{u}}+\dot{J}\mathbf{F}^{-1}\mathbf{u}+J\partial(\mathbf{F}^{-1})/\partial t\,\mathbf{u}]\cdot\mathbf{N}dA.
\end{aligned}$$

From (3.43) we have $\dot{J} = J\mathrm{tr}\,(\mathbf{L})$, and from (3.45) we have $\partial(\mathbf{F}^{-1})/\partial t = -\mathbf{F}^{-1}\mathbf{L}$. Thus,

$$\begin{aligned}
\tfrac{d}{dt}\textstyle\int_{S_t}\mathbf{u}\cdot\mathbf{n}da &= \textstyle\int_{S_r}[J\mathbf{F}^{-1}\dot{\mathbf{u}}+J\mathrm{tr}\,(\mathbf{L})\mathbf{F}^{-1}\mathbf{u}-J\mathbf{F}^{-1}\mathbf{L}\mathbf{u}]\cdot\mathbf{N}dA \\
&= \textstyle\int_{S_r}\{J\mathbf{F}^{-1}[\dot{\mathbf{u}}+\mathrm{tr}\,(\mathbf{L})\mathbf{u}-\mathbf{L}\mathbf{u}]\}\cdot\mathbf{N}dA \\
&= \textstyle\int_{S_r}[\dot{\mathbf{u}}+\mathrm{tr}\,(\mathbf{L})\mathbf{u}-\mathbf{L}\mathbf{u}]\cdot(J\mathbf{F}^{-T}\mathbf{N})dA \\
&= \textstyle\int_{S_t}[\dot{\mathbf{u}}+\mathrm{tr}\,(\mathbf{L})\mathbf{u}-\mathbf{L}\mathbf{u}]\cdot\mathbf{n}da \\
&\qquad \text{(converting back to an integral over } S_t\text{)}.
\end{aligned}$$

We can establish the other formulae by following the same approach.