2 Cartesian tensors

2.1 Change of basis

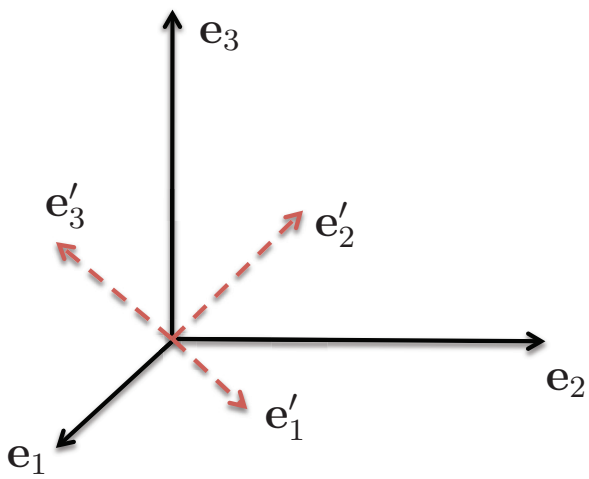
Let $\mathscr{V}$ denote a three-dimensional (Euclidean) vector space, and select right-handed orthonormal basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ for $\mathscr{V}$ so that

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases} \tag{2.1}$$

Then, in the summation convention, any vector $\mathbf{v}$ can be written as

$$\mathbf{v} = v_i \mathbf{e}_i \tag{2.2}$$

with respect to the chosen basis.



Now choose a second orthonormal basis $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$. Again, we have the connections

$$\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}, \tag{2.3}$$

and we may also decompose $\mathbf{v}$ with respect to this new basis as

$$\mathbf{v} = v'_i \mathbf{e}'_i. \tag{2.4}$$

Since $\mathscr{V}$ has three dimensions, $\mathbf{e}'_i$ must be expressible as a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ for each $i \in \{1, 2, 3\}$. We write this in the form

$$\mathbf{e}'_i = l_{ij} \mathbf{e}_j \qquad (i \in \{1, 2, 3\};\ \text{sum over } j). \tag{2.5}$$

It follows from (2.5) that

$$l_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j, \tag{2.6}$$

which are the so-called *direction cosines* of $\mathbf{e}'_i$ relative to $\mathbf{e}_j$.

From (2.3) and (2.5) we have

$$\delta_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j = (l_{ik}\mathbf{e}_k) \cdot \mathbf{e}'_j = l_{ik}(\mathbf{e}'_j \cdot \mathbf{e}_k) = l_{ik}l_{jk},$$

and similarly

$$l_{ki}l_{kj} = \delta_{ij}.$$

Thus

$$l_{ik}l_{jk} = \delta_{ij} = l_{ki}l_{kj}. \tag{2.7}$$

In matrix notation, defined by $L \equiv (l_{ij})_{i,j=1,2,3} \in M_{3\times 3}(\mathbb{R})$, $I \equiv (\delta_{ij})_{i,j=1,2,3} \in M_{3\times 3}(\mathbb{R})$, equations (2.7) may be expressed as

$$LL^T = I = L^T L, \tag{2.8}$$

from which it follows that $\det(LL^T) = 1$ and hence $\det L = \pm 1$. Thus, L is an *orthogonal* matrix. If $\det L = +1$ then L is said to be *proper orthogonal*, and it represents a rotation, while if $\det L = -1$ then L is said to be *improper orthogonal*, and it represents a rotation combined with a reflection.

We now have

$$l_{ik}\mathbf{e}'_i = l_{ik}l_{ij}\mathbf{e}_j = \delta_{jk}\mathbf{e}_j = \mathbf{e}_k,$$

and, combining this with (2.5), we obtain the connections

$$\mathbf{e}'_i = l_{ij}\mathbf{e}_j, \quad \mathbf{e}_i = l_{ji}\mathbf{e}'_j. \tag{2.9}$$

Also, since

$$\mathbf{v} = v_j\mathbf{e}_j = v_j l_{ij}\mathbf{e}'_i, \quad \mathbf{v} = v'_i\mathbf{e}'_i$$

we obtain $v'_i = l_{ij}v_j$, and, similarly, $v_i = l_{ji}v'_j$. Thus, *the components of a vector transform in the same way as the basis vectors under a change of basis* — compare (2.9) with

$$v'_i = l_{ij}v_j, \quad v_i = l_{ji}v'_j. \tag{2.10}$$

Examples

1. If the origin is fixed then the components of the position vector $\mathbf{x}$ transform according to

$$x'_i = l_{ij}x_j, \quad x_j = l_{ij}x'_i,$$

and it follows that

$$l_{ij} = \frac{\partial x'_i}{\partial x_j} = \frac{\partial x_j}{\partial x'_i}.$$

2. The transformation of basis vectors defined by

$$L = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

leaves $\mathbf{e}_3$ unchanged and represents an anticlockwise rotation through an angle θ in the $(1,2)$-plane.

3. The transformation of basis vectors defined by

$$L = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & -\cos 2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

represents a rotation through an angle 2α about $\mathbf{e}_3$ followed by a reflection in $\mathbf{e}'_1$ in the $(1,2)$-plane.

2.2 Definition of a Cartesian tensor

The following definition relates to the three-dimensional space $\mathscr{V}$, but is equally applicable to vector spaces of higher dimension.

A *second-order tensor* $\mathbf{T}$ is a linear mapping $\mathbf{T} : \mathscr{V} \to \mathscr{V}$.

If we consider V in conjunction with the set of all possible right-handed orthonormal basis vectors related by (2.9) with L *proper orthogonal* then $\mathbf{T}$ is referred to as a *Cartesian tensor*, and its components T_{ij} and T'_{ij} relative to the bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ respectively are related by

$$T'_{ij} = l_{ip} l_{jq} T_{pq}. \tag{2.11}$$

The transformation rule (2.11) can be established as follows. Let $\mathbf{u} \in \mathscr{V}$ and let $\mathbf{v} = \mathbf{T}\mathbf{u} \in \mathscr{V}$, so that

$$v_i = T_{ij} u_j, \quad v'_i = T'_{ij} u'_j$$

with respect to bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ respectively. But, since $\mathbf{u}$ and $\mathbf{v}$ are vectors, we have

$$v'_i = l_{ip} v_p, \quad u'_j = l_{jq} u_q.$$

Hence

$$l_{ip} T_{pq} u_q = l_{ip} v_p = T'_{ij} l_{jq} u_q,$$

which may be re-written as

$$(l_{ip} T_{pq} - T'_{ij} l_{jq}) u_q = 0.$$

This must hold for all choices of u_q. Hence

$$l_{ip}T_{pq} - T'_{ip}l_{pq} = 0.$$

Multiplication of this by l_{jq} then gives

$$l_{ip}l_{jq}T_{pq} - T'_{ip}(l_{pq}l_{jq}) = 0,$$

i.e.

$$T'_{ij} = l_{ip}l_{jq}T_{pq},$$

as required.

In matrix notation, we write

$$[\mathbf{T}]' = L[\mathbf{T}]L^T.$$

Analogously to (2.11), a *Cartesian tensor of order* n – in brief CT(n) – has components $T_{ijk\ldots}$ (with n indices $i, j, k, \ldots$) which transform according to the rule

$$T'_{ijk\ldots} = l_{ip}l_{jq}l_{kr}\ldots T_{pqr\ldots} \tag{2.12}$$

under a change of orthonormal basis. [More generally, $\mathbf{T}$ is referred to as *multilinear mapping* over $\mathscr{V}$, but we shall not use this terminology in this course.]

Examples

1. A scalar is CT(0): $\alpha' = \alpha$.

2. A vector is CT(1): $v'_i = l_{ij}v_j$.

3. The *dyadic* (or *tensor*) *product* of two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \otimes \mathbf{v}$, is CT(2). Its components are defined by $(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i v_j$, and the transformation rule is simply

$$(\mathbf{u} \otimes \mathbf{v})'_{ij} = u'_i v'_j = l_{ip}u_p l_{jq}v_q = l_{ip}l_{jq}u_p v_q = l_{ip}l_{jq}(\mathbf{u} \otimes \mathbf{v})_{pq}.$$

4. The Kronecker delta is CT(2):

$$\begin{aligned} \delta'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j &= (l_{ip}\mathbf{e}_p) \cdot (l_{jq}\mathbf{e}_q) \\ &= l_{ip}l_{jq}(\mathbf{e}_p \cdot \mathbf{e}_q) \\ &= l_{ip}l_{jq}\delta_{pq} \ (= \delta_{ij}) \end{aligned}$$

Note that the components of the Kronecker delta are the same for *all* (orthonormal) bases. The Kronecker delta is an example of an *isotropic tensor*. Isotropic tensors will be discussed later.

2.3 The tensor product notation

The tensor product notation has been defined in Example 3 above. An alternative definition (without reference to choice of basis) is as follows. For a given pair of vectors $\mathbf{u}$ and $\mathbf{v}$ the tensor product $\mathbf{u} \otimes \mathbf{v}$ is defined by the identity

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{a} = (\mathbf{v} \cdot \mathbf{a})\mathbf{u} \qquad \text{for all } \mathbf{a} \in \mathscr{V}. \tag{2.13}$$

The component form of (2.13) with respect to the basis $\{\mathbf{e}_i\}$ is

$$(\mathbf{u} \otimes \mathbf{v})_{ij} a_j = v_j a_j u_i \qquad \text{for all } \mathbf{a} \in \mathscr{V},$$

and this is equivalent to the definition given in Example 3.

Important note: in general, a CT(2) *cannot* be represented as the tensor product of two vectors, i.e. a given $\mathbf{T}$ cannot be written in the form $\mathbf{u} \otimes \mathbf{v}$ for any pair of vectors $\mathbf{u}, \mathbf{v} \in \mathscr{V}$. However, $\mathbf{T}$ can always be written as a linear combination of products of the form $\mathbf{u} \otimes \mathbf{v}$; the reason for this is as follows.

Note that if α and β are scalars and $\mathbf{S}$ and $\mathbf{T}$ are CT(2) then $\alpha\mathbf{S} + \beta\mathbf{T}$ is also CT(2). It follows that the set of all CT(2)s forms a vector space (*not* the same as $\mathscr{V}$). This space has dimension $9 = 3^2$ (note that T_{ij} has 9 components — $T_{11}, T_{12}, T_{13}, \ldots, T_{33}$). A basis 'vector' for this 9-dimensional space can be written as $\mathbf{e}_i \otimes \mathbf{e}_j$ — there are 9 such objects altogether:

$$\begin{array}{ccc} \mathbf{e}_1 \otimes \mathbf{e}_1 & \mathbf{e}_1 \otimes \mathbf{e}_2 & \mathbf{e}_1 \otimes \mathbf{e}_3 \\ \mathbf{e}_2 \otimes \mathbf{e}_1 & \mathbf{e}_2 \otimes \mathbf{e}_2 & \mathbf{e}_2 \otimes \mathbf{e}_3 \\ \mathbf{e}_3 \otimes \mathbf{e}_1 & \mathbf{e}_3 \otimes \mathbf{e}_2 & \mathbf{e}_3 \otimes \mathbf{e}_3. \end{array}$$

Note, in particular, that $\mathbf{e}_1 \otimes \mathbf{e}_2 \neq \mathbf{e}_2 \otimes \mathbf{e}_1$.

Thus, $\mathbf{T}$ may be written in the form

$$\begin{aligned} \mathbf{T} &= T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \qquad \text{(sum over } i \text{ and } j\text{)} \\ &\equiv T_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + T_{12}\mathbf{e}_1 \otimes \mathbf{e}_2 + T_{13}\mathbf{e}_1 \otimes \mathbf{e}_3 \\ &\quad + T_{21}\mathbf{e}_2 \otimes \mathbf{e}_1 + T_{22}\mathbf{e}_2 \otimes \mathbf{e}_2 + T_{23}\mathbf{e}_2 \otimes \mathbf{e}_3 \\ &\quad + T_{31}\mathbf{e}_3 \otimes \mathbf{e}_1 + T_{32}\mathbf{e}_3 \otimes \mathbf{e}_2 + T_{33}\mathbf{e}_3 \otimes \mathbf{e}_3 \end{aligned} \tag{2.14}$$

If we choose a second basis $\mathbf{e}'_i \otimes \mathbf{e}'_j$ then we may also write

$$\mathbf{T} = T'_{ij}\mathbf{e}'_i \otimes \mathbf{e}'_j. \tag{2.15}$$

Consider the operation of $\mathbf{T}$ on a vector $\mathbf{u}$:

$$\begin{aligned} \mathbf{T}\mathbf{u} &= (T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{u} && \text{by (2.14)} \\ &= T_{ij}(\mathbf{u} \cdot \mathbf{e}_j)\mathbf{e}_i && \text{by (2.13)} \\ &= T_{ij}u_j\mathbf{e}_i. \end{aligned}$$

Thus,

$$(\mathbf{Tu})_i = T_{ij}u_j.$$

More particularly,

$$\begin{aligned}\mathbf{Te}_j &= (T_{pq}\mathbf{e}_p \otimes \mathbf{e}_q)\mathbf{e}_j \\ &= T_{pq}(\mathbf{e}_q \cdot \mathbf{e}_j)\mathbf{e}_p \\ &= T_{pq}\delta_{qj}\mathbf{e}_p = T_{pj}\mathbf{e}_p\end{aligned}$$

and hence

$$\mathbf{e}_i \cdot (\mathbf{Te}_j) = T_{pj}\mathbf{e}_i \cdot \mathbf{e}_p = T_{pj}\delta_{ip} = T_{ij}.$$

For a given $\mathbf{T}$ and choice of basis $\{\mathbf{e}_i\}$, therefore, we may *define* the components T_{ij} by

$$T_{ij} = \mathbf{e}_i \cdot (\mathbf{Te}_j) \equiv (\mathbf{Te}_j) \cdot \mathbf{e}_i. \tag{2.16}$$

2.4 The transpose of a CT(2)

The *transpose* $\mathbf{T}^T$ of $\mathbf{T}$ is defined via the identity

$$(\mathbf{Tu}) \cdot \mathbf{v} \equiv (\mathbf{T}^T\mathbf{v}) \cdot \mathbf{u} \qquad \text{for all } \mathbf{u}, \mathbf{v} \in \mathscr{V}. \tag{2.17}$$

Taking $\mathbf{u} = \mathbf{e}_j, \mathbf{v} = \mathbf{e}_i$ in (2.17) and using (2.16) we deduce that

$$T_{ij} = (\mathbf{T}^T)_{ji},$$

as might be expected. Note that

$$(\mathbf{a} \otimes \mathbf{b})^T = \mathbf{b} \otimes \mathbf{a}.$$

Definition

$\mathbf{T}$ is *symmetric* if $\mathbf{T}^T = \mathbf{T}$

$\mathbf{T}$ is *skewsymmetric* or *antisymmetric* if $\mathbf{T}^T = -\mathbf{T}$

Note that a symmetric CT(2) has 6 independent components and a skewsymmetric CT(2) has 3 independent components, and that any CT(2) $\mathbf{T}$ can be written as the sum of a symmetric and an antisymmetric part:

$$\mathbf{T} = \frac{1}{2}(\mathbf{T} + \mathbf{T}^T) + \frac{1}{2}(\mathbf{T} - \mathbf{T}^T). \tag{2.18}$$

2.5 The axial vector associated with a skewsymmetric CT(2)

If $\mathbf{W}$ is a skewsymmetric CT(2) then there exists a vector $\mathbf{w} \in \mathscr{V}$ (called the *axial vector* of $\mathbf{W}$) such that

$$\mathbf{Wv} = \mathbf{w} \times \mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathscr{V}. \tag{2.19}$$

Proof. Since $\mathbf{W}$ is skewsymmetric ($W_{ij} = -W_{ji}$) it may be written

$$\mathbf{W} = W_{23}(\mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_2) + W_{13}(\mathbf{e}_1 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_1) + W_{12}(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1).$$

Then, using (2.13), we obtain

$$\begin{aligned}\mathbf{W}\mathbf{v} &= W_{23}v_3\mathbf{e}_2 - W_{23}v_2\mathbf{e}_3 + W_{13}v_3\mathbf{e}_1 - W_{13}v_1\mathbf{e}_3 + W_{12}v_2\mathbf{e}_1 - W_{12}v_1\mathbf{e}_2 \\ &= (W_{13}v_3 + W_{12}v_2)\mathbf{e}_1 + (-W_{12}v_1 + W_{23}v_3)\mathbf{e}_2 + (-W_{23}v_2 - W_{13}v_1)\mathbf{e}_3.\end{aligned}$$

By writing $w_1 = -W_{23}, w_2 = W_{13}, w_3 = -W_{12}$ this becomes

$$\begin{aligned}\mathbf{W}\mathbf{v} &= (w_2v_3 - w_3v_2)\mathbf{e}_1 + (w_3v_1 - w_1v_3)\mathbf{e}_2 + (w_1v_2 - w_2v_1)\mathbf{e}_3 \\ &\equiv \mathbf{w} \times \mathbf{v}.\end{aligned}$$

2.6 Scalar product of a CT(2)

The vector space CT(2) has a natural *scalar product*:

$$\mathbf{S} \cdot \mathbf{T} \equiv \mathrm{tr}(\mathbf{S}^T\mathbf{T}) = \mathrm{tr}(\mathbf{S}\mathbf{T}^T) = S_{ij}T_{ij}.$$

It is straightforward to prove that

(a) if $\mathbf{S}$ is symmetric,

$$\mathbf{S} \cdot \mathbf{T} = \mathbf{S} \cdot \mathbf{T}^T = \mathbf{S} \cdot \left[\frac{1}{2}(\mathbf{T} + \mathbf{T}^T)\right];$$

(b) if $\mathbf{W}$ is skew-symmetric,

$$\mathbf{W} \cdot \mathbf{T} = -\mathbf{W} \cdot \mathbf{T}^T = \mathbf{W} \cdot \left[\frac{1}{2}(\mathbf{T} - \mathbf{T}^T)\right];$$

(c) if $\mathbf{S}$ is symmetric and $\mathbf{W}$ is skew-symmetric,

$$\mathbf{S} \cdot \mathbf{T} = 0;$$

(d) if $\mathbf{T} \cdot \mathbf{S} = 0$ for every $\mathbf{S} \in$CT(2), then $\mathbf{T} = \mathbf{O}$;

(e) if $\mathbf{T} \cdot \mathbf{S} = 0$ for every symmetric tensor $\mathbf{S} \in$CT(2), then $\mathbf{T}$ is skew-symmetric;

(f) if $\mathbf{T} \cdot \mathbf{S} = 0$ for every tensor skew-symmetric $\mathbf{S} \in$CT(2), then $\mathbf{T}$ is symmetric.

A second-order Cartesian tensor $\mathbf{T}$ is *invertible* if there exists a tensor $\mathbf{T}^{-1}$, called the *inverse* of $\mathbf{T}$, such that

$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{I} = \mathbf{T}^{-1}\mathbf{T}.$$

It can be proved that $\mathbf{T}$ is invertible if and only if $\det \mathbf{T} \neq 0$.

The identities

$$\begin{aligned}
\det(\mathbf{ST}) &= \det \mathbf{S} \det \mathbf{T}, \\
\det \mathbf{T}^T &= \det \mathbf{T}, \\
\det(\mathbf{T}^{-1}) &= (\det \mathbf{T})^{-1}, \\
(\mathbf{ST})^{-1} &= \mathbf{T}^{-1}\mathbf{S}^{-1}, \\
(\mathbf{T}^{-1})^T &= (\mathbf{T}^T)^{-1}
\end{aligned}$$

will be useful. For convenience, we use the abbreviation

$$\mathbf{S}^{-T} \equiv (\mathbf{S}^{-1})^T.$$

A tensor $\mathbf{Q} \in$CT(2) is *orthogonal* if it preserves the scalar products:

$$\mathbf{Qu} \cdot \mathbf{Qv} = \mathbf{u} \cdot \mathbf{v} \quad \forall \mathbf{u}, \mathbf{v} \in \mathscr{V}. \tag{2.20}$$

The following proposition gives a characterization of orthogonal tensors.
Proposition 1. *A necessary and sufficient condition that* $\mathbf{Q}$ *be orthogonal is that*

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I}.$$

A second-order tensor $\mathbf{T}$ is *positive definite* provided

$$\mathbf{u} \cdot \mathbf{Tu} > 0 \quad \forall \mathbf{u} \in \mathscr{V} - \{\mathbf{0}\}.$$

Throughout this course we will use the following notation

$$\begin{aligned}
&\mathrm{Lin}^+ = \{\mathbf{T} \in \mathrm{CT}(2) : \det \mathbf{T} > 0\}, \\
&\mathrm{Sym} = \{\mathbf{T} \in \mathrm{CT}(2) : \mathbf{T}^T = \mathbf{T}\}, \\
&\mathrm{Skw} = \{\mathbf{T} \in \mathrm{CT}(2) : \mathbf{T}^T = -\mathbf{T}\}, \\
&\mathrm{Psym} = \{\mathbf{T} \in \mathrm{Sym} : \mathbf{T} \text{ positive definite}\}, \\
&\mathrm{Orth} = \{\mathbf{T} \in \mathrm{CT}(2) : \mathbf{T} \text{ orthogonal}\}, \\
&\mathrm{Orth}^+ = \{\mathbf{T} \in \mathrm{Orth} : \det \mathbf{T} = 1\}.
\end{aligned}$$

2.7 Contraction of tensors

Let $\mathbf{T}$ be a CT(n) with components $T_{ijk\ldots pq\ldots}$ with respect to the basis $\{\mathbf{e}_i\}$. Set $q = k$ and sum over k. These indices are then said to be *contracted*, and the effect of this operation is to produce a CT($n-2$). This procedure is illustrated by the following examples.

Examples

1. Suppose T_{ij} are the components of a CT(2), so that, under a transformation of basis, they change according to

$$T'_{ij} = l_{ip} l_{jq} T_{pq}.$$

Contraction of i with j gives

$$T'_{ii} = l_{ip} l_{iq} T_{pq} = \delta_{pq} T_{pq} = T_{pp} = T_{ii}.$$

Thus, T_{ii} is a CT(0). In fact, it is an example of a *scalar invariant* of $\mathbf{T}$, called the *trace* of $\mathbf{T}$, denoted $\mathrm{tr}(\mathbf{T})$.

2. Under contraction the tensor product $\mathbf{u} \otimes \mathbf{v}$ becomes the scalar product $\mathbf{u} \cdot \mathbf{v}$ since the components $u_i v_j$ contract to $u_i v_i$.

3. If T_{ijkl} are the components of a CT(4) then

$$T'_{ijkl} = l_{ip} l_{jq} l_{kr} l_{ls} T_{pqrs}.$$

Contraction of the indices k and l gives

$$T'_{ijkk} = l_{ip} l_{jq} \delta_{rs} T_{pqrs} = l_{ip} l_{jq} T_{pqrr},$$

so that T_{ijkk} are the components of a CT(2). [The proof in the general case is similar.]

2.8 The determinant of a CT(2)

Recall from Section 0 that if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathscr{V}$ then their triple scalar product may be written as a determinant

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

or, equivalently, in terms of the alternating symbol, as

$$\epsilon_{pqr} u_p v_q w_r = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}.$$

It follows that if T_{ij} are the components of a CT(2) then

$$\epsilon_{pqr} T_{pi} T_{qj} T_{rk} = \begin{vmatrix} T_{1i} & T_{1j} & T_{1k} \\ T_{2i} & T_{2j} & T_{2k} \\ T_{3i} & T_{3j} & T_{3k} \end{vmatrix}.$$

By considering different values of i, j, k we may deduce that

$$\begin{vmatrix} T_{1i} & T_{1j} & T_{1k} \\ T_{2i} & T_{2j} & T_{2k} \\ T_{3i} & T_{3j} & T_{3k} \end{vmatrix} = [\det(T_{mn})] \epsilon_{ijk},$$

where

$$\det(T_{mn}) = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix}.$$

Hence

$$\epsilon_{pqr} T_{pi} T_{qj} T_{rk} = [\det(T_{mn})]\epsilon_{ijk}. \tag{2.21}$$

In particular, by taking $i = 1, j = 2, k = 3$, we obtain an expression for $\det(T_{mn})$, namely

$$\det(T_{mn}) = \epsilon_{pqr} T_{p1} T_{q2} T_{r3}. \tag{2.22}$$

An alternative expression for $\det(T_{mn})$ can be obtained by multiplying (2.20) by ϵ_{ijk} and noting that

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \quad \epsilon_{ijk}\epsilon_{ijk} = 6,$$

to give

$$\det(T_{mn}) = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr} T_{pi} T_{qj} T_{rk}. \tag{2.23}$$

Important example

The result (2.21) applies for any matrix (T_{ij}). In particular, application to (l_{ij}) and use of the fact that (l_{ij}) is proper orthogonal, so that $\det(l_{ij}) = 1$, leads to

$$\epsilon_{ijk} = \epsilon_{pqr} l_{pi} l_{qj} l_{rk}. \tag{2.24}$$

This indicates that ϵ_{ijk} are the components of a third-order *isotropic* tensor — see Section 2.9 for further discussion of isotropic tensors.

Example

We show that $\det(T_{ij})$ is independent of the choice of (orthonormal) basis.

$$\begin{aligned}
\det(T'_{ij}) &= \epsilon'_{pqr} T'_{p1} T'_{q2} T'_{r3} \qquad \text{using (2.22)} \\
&= \epsilon_{pqr}(l_{pi} l_{1j} T_{ij})(l_{qk} l_{2l} T_{kl})(l_{rm} l_{3n} T_{mn}) \\
&\qquad \text{using (2.24) and (2.12)} \\
&= \epsilon_{pqr} l_{pi} l_{qk} l_{rm} l_{1j} l_{2l} l_{3n} T_{ij} T_{kl} T_{mn} \\
&= \epsilon_{ikm} T_{ij} T_{kl} T_{mn} l_{1j} l_{2l} l_{3n} \\
&\qquad \text{using (2.24) and rearranging} \\
&= [\det(T_{ij})]\epsilon_{jln} l_{1j} l_{2l} l_{3n} \qquad \text{using (2.21)} \\
&= \det(T_{ij})\epsilon_{123} \qquad \text{(using (2.24))} \\
&= \det(T_{ij}).
\end{aligned}$$

This allows us to define the *determinant*, $\det \mathbf{T}$, of a CT(2) as $\det(T_{ij})$ for any choice of (orthonormal) basis. It is a *scalar invariant* of $\mathbf{T}$.

2.9 Isotropic tensors

Definition: if the components of a CT(n) $\mathbf{T}$ are unchanged under arbitrary *rotations* of orthonormal bases then $\mathbf{T}$ is said to be *isotropic*.

Examples

1. CT(0): all scalars are isotropic.

2. CT(1): $\mathbf{0}$ is the only isotropic vector.

This may be proved as follows. To satisfy the definition of isotropy we require

$$l_{ij}u_j \equiv u'_i = u_i \qquad \text{for all rotations } (l_{ij}).$$

We are free to choose l_{ij} to be any rotation. The choice

$$l_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(representing a rotation of $\pi/2$ about $\mathbf{e}_3$) leads to $u_1 = u_2 = 0$. Any other choice leads immediately to $u_3 = 0$ and hence $\mathbf{u} = \mathbf{0}$.

3. CT(2): scalar multiples of δ_{ij} are the only isotropic CT(2).

For isotropy we require

$$l_{ip}l_{jq}u_{pq} \equiv u'_{ij} = u_{ij} \qquad \text{for all rotations } (l_{ij}).$$

The same choice as in Example 2 gives $u_{11} = u_{22}, u_{13} = u_{23} = 0$. Choosing (l_{ij}) to represent a rotation through $\pi/2$ about $\mathbf{e}_1$ instead of $\mathbf{e}_3$ gives $u_{22} = u_{33}, u_{21} = u_{31} = 0$ etc. and we finish up with $u_{ij} = u_{11}\delta_{ij}$, where u_{11} is an arbitrary scalar.

4. CT(3): scalar multiples of ϵ_{ijk} are the only isotropic CT(3).

This can be proved in a similar way to the proof given in Example 2 (details are omitted since they are a bit tedious).

5. CT(4): the only independent isotropic CT(4)s with free indices i, j, k, l are scalar multiples of

$$\delta_{ij}\delta_{kl}, \quad \delta_{ik}\delta_{jl}, \quad \delta_{il}\delta_{jk}$$

and the most general isotropic CT(4) has components of the form

$$\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk},$$

where α, β, γ are arbitrary scalars (the proof follows the same pattern as before and is omitted).

6. We use the result of Example 5 to prove the identity

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}.$$

Since ϵ_{ijk} is isotropic CT(3) it follows that $\epsilon_{ijk}\epsilon_{pqr}$ is isotropic CT(6) and hence, on contraction, we deduce that $\epsilon_{ijk}\epsilon_{pqk}$ is isotropic CT(4). But, the most general isotropic CT(4) has the form given in Example 5, so we can write

$$\epsilon_{ijk}\epsilon_{pqk} = \alpha\delta_{ij}\delta_{pq} + \beta\delta_{ip}\delta_{jq} + \gamma\delta_{iq}\delta_{jp}.$$

Setting $ij = 11, pq = 22$ in the above leads to $\alpha = 0$. Next, setting $ij = pq = 11$ gives $\beta + \gamma = 0$ and finally $ij = pq = 12$ gives $\beta = 1$. Hence the identity is established.

7. For tensors of higher order than 4 we note here that if n is even then the only isotropic CT(n)s are linear combinations of products of Kronecker deltas, while if n is odd the only isotropic CT(n)s are linear combinations of the products of one alternating symbol and an appropriate number of Kronecker deltas.

2.10 Eigenvalues and eigenvectors of a second-order tensor

Let $\mathbf{T}$ be a second-order tensor. A scalar λ is an *eigenvalue* of $\mathbf{T}$ if there exists a non-zero vector $\mathbf{u} \in \mathscr{V}$ such that

$$\mathbf{T}\mathbf{u} = \lambda\mathbf{u}, \tag{2.25}$$

in which case $\mathbf{u}$ is called an *eigenvector* of $\mathbf{T}$ corresponding to the eigenvalue λ.

The set of homogeneous equations (2.25) has non-trivial solutions if and only if

$$\det(\mathbf{T} - \lambda\mathbf{I}) = 0. \tag{2.26}$$

This is called the *characteristic equation* for $\mathbf{T}$, which is a third-degree algebraic equation in λ. Therefore any second-order tensor has at least one *real* eigenvalue and at most three *real* eigenvalues. The list of the eigenvalues of $\mathbf{T}$ in which each eigenvalue is repeated a number of times equal to its algebraic multiplicity is called the *spectrum* of $\mathbf{T}$.

Proposition 2. *The eigenvalues of a positive definite second-order tensor are strictly positive.*

Proof. Let λ be an eigenvalue of a positive definite tensor $\mathbf{T}$, and let $\mathbf{u}$ be a corresponding eigenvector, *i.e.* $\mathbf{T}\mathbf{u} = \lambda\mathbf{u}$. Then, since $\mathbf{u} \neq \mathbf{0}$, $\lambda|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{T}\mathbf{u} > 0$ and thus $\lambda > 0$. □

Expansion of the determinant (2.26) leads to the equation

$$\lambda^3 - I_1(\mathbf{T})\lambda^2 + I_2(\mathbf{T})\lambda - I_3(\mathbf{T}) = 0, \tag{2.27}$$

where

$$\left.\begin{aligned} I_1(\mathbf{T}) &= \mathrm{tr}\mathbf{T}, \\ I_2(\mathbf{T}) &= \frac{1}{2}\left[(\mathrm{tr}\mathbf{T})^2 - \mathrm{tr}\mathbf{T}^2\right], \\ I_3(\mathbf{T}) &= \det \mathbf{T}. \end{aligned}\right\} \tag{2.28}$$

These quantities are called the *principal invariants* of $\mathbf{T}$. This terminology follows from the invariance of the trace and the determinant of a second-order tensor under change of basis of $\mathscr{V}$.

Cayley-Hamilton theorem: *Every second-order tensor* $\mathbf{T}$ *satisfies its own characteristic equation*

$$\mathbf{T}^3 - I_1(\mathbf{T})\mathbf{T}^2 + I_2(\mathbf{T})\mathbf{T} - I_3(\mathbf{T})\mathbf{I} = \mathbf{O}. \tag{2.29}$$

2.11 Symmetric second-order tensors

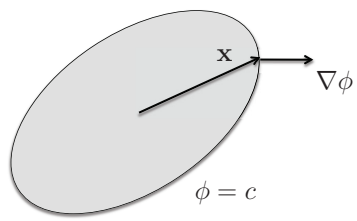
For symmetric second-order tensors the results of the previous section can be made more specific. Let $\mathbf{T}$ be a *symmetric* CT(2). Consider the quadratic form

$$\begin{aligned} \phi(\mathbf{x}) &\equiv (\mathbf{T}\mathbf{x}) \cdot \mathbf{x} \equiv T_{ij}x_jx_i \\ &\equiv T_{11}x_1^2 + \cdots + 2T_{12}x_1x_2 + \ldots \end{aligned} \tag{2.30}$$

The equation

$$\phi(\mathbf{x}) = c, \tag{2.31}$$

where c is a constant, defines a quadratic surface (ellipsoid, paraboloid or hyperboloid). For example, if $T_{11} = 1/a^2$, $T_{22} = 1/b^2$, $T_{33} = 1/c^2$, $T_{12} = T_{23} = T_{13} = 0$ and the c in (2.31) is 1, then we have an ellipsoid with semi-axes a, b, c. The vector $\nabla\phi(\mathbf{x})$ is normal



to the surface (2.31) at the point $\mathbf{x}$ and from (2.30) we calculate

$$\nabla\phi(\mathbf{x}) = 2\mathbf{T}\mathbf{x}.$$

When the position vector $\mathbf{x}$ of a point P on the surface (2.31) is parallel to the normal at P then the direction OP is said to be a *principal direction* (or principal axis) of the quadratic surface, and hence a principal direction (or axis) or **eigenvector** of $\mathbf{T}$. This requires that $\nabla\phi(\mathbf{x})$ is proportional to $\mathbf{x}$, and hence

$$\mathbf{T}\mathbf{x} = \lambda\mathbf{x} \quad (T_{ij}x_j = \lambda x_i \text{ in index notation}), \tag{2.32}$$

where λ is a scalar—the *principal value* (or **eigenvalue**) associated with the *principal direction* (or **eigenvector**) $\mathbf{x}$.

Theorem

(a) There exist three mutually orthogonal eigenvectors, $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}$ say,

(b) the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are real,

(c) orthonormal basis vectors $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$, coincident with $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}$ can be chosen so that $\mathbf{T}$ has components $T'_{ij} = \lambda_i \delta_{ij}$ (no summation), i.e.

$$\mathbf{T} = \sum_{i=1}^{3} \lambda_i \mathbf{e}'_i \otimes \mathbf{e}'_i. \tag{2.33}$$

(c) is also known as the Spectral Representation.

Proof: from (2.32) we have

$$(\mathbf{T} - \lambda_i \mathbf{I})\mathbf{x}^{(i)} = \mathbf{0} \quad i \in \{1, 2, 3\}.$$

Choose axes $\mathbf{e}'_i$ coincident with $\mathbf{x}^{(i)}$, so that

$$\mathbf{T}\mathbf{e}'_i = \lambda_i \mathbf{e}'_i \quad \text{(no sum over } i\text{)}.$$

Next, take the dot product of this equation with $\mathbf{e}'_j$:

$$T'_{ij} \equiv T'_{ji} = \mathbf{e}'_j \cdot (\mathbf{T}\mathbf{e}'_i) = \lambda_i \mathbf{e}'_j \cdot \mathbf{e}'_i = \lambda_i \delta_{ij}.$$

Thus,

$$\begin{aligned} \mathbf{T} &= T'_{ij} \mathbf{e}'_i \otimes \mathbf{e}'_j \\ &= \lambda_i \delta_{ij} \mathbf{e}'_i \otimes \mathbf{e}'_j \quad \text{(sum over } i \text{ and } j\text{)} \\ &= \textstyle\sum_{i=1}^{3} t_i \mathbf{e}'_i \otimes \mathbf{e}'_i. \end{aligned}$$

2.12 Higher-order tensors

Extended tensor product notation A CT(n) can be represented in the form

$$\mathbf{T} = T_{ijk\ldots} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \ldots$$

The tensor product of a CT(m), $\mathbf{S}$, and a CT(n), $\mathbf{T}$, is a CT($m+n$) defined by

$$\begin{aligned} \mathbf{S} \otimes \mathbf{T} &\equiv (S_{ijk\ldots} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots) \otimes (T_{pqr\ldots} \mathbf{e}_p \otimes \mathbf{e}_q \otimes \ldots) \\ &= S_{ijk\ldots} T_{pqr\ldots} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \cdots \otimes \mathbf{e}_p \otimes \mathbf{e}_q \otimes \ldots \end{aligned}$$

Note: there is no need to distinguish between, for example, $(\mathbf{e}_i \otimes \mathbf{e}_j) \otimes \mathbf{e}_k$ and $\mathbf{e}_i \otimes (\mathbf{e}_j \otimes \mathbf{e}_k)$, and we therefore write $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$, omitting the brackets.

Note: if $\mathbf{S}$ and $\mathbf{T}$ are both CT(n)s and α, β are real numbers then $\alpha\mathbf{S} + \beta\mathbf{T}$ is also CT(n). CT(n)s therefore form a vector space of dimension 3^n with basis 'vectors' $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \ldots$

3 Kinematics

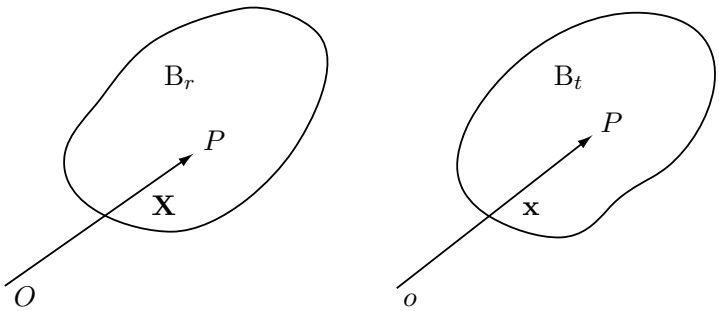
3.1 Bodies, configurations and motions

Definition: A *body* $\mathcal{B}$ is a set whose elements can be put into one-to-one correspondence with points of a region B in three-dimensional Euclidean point space.

The elements of $\mathcal{B}$ are called *particles* (or *material points*), and B is called a *configuration* of $\mathcal{B}$.

As the body moves the configuration changes with time. Let $t \in I \subset \mathbb{R}$ denote time, where I is an interval in $\mathbb{R}$. If, with each $t \in I$, we associate a unique configuration B_t of $\mathcal{B}$ then the family of configurations $\{\mathrm{B}_t : t \in I\}$ is called a *motion* of $\mathcal{B}$. We assume that as $\mathcal{B}$ moves continuously then B_t changes continuously.

It is convenient to identify a *reference configuration*, B_r say, which is an arbitrarily chosen fixed configuration. Then, any particle P of $\mathcal{B}$ may be labelled by its position vector $\mathbf{X}$ in B_r relative to some origin O. Let $\mathbf{x}$ be the position vector of P in the configuration B_t at time t relative to an origin o (which need not coincide with O).



We say that $\mathcal{B}$ *occupies* the configuration B_t at time t — B_t is also referred to as the *current configuration*.

[Note that B_r need not be a configuration actually occupied by $\mathcal{B}$ during the motion, but is often chosen to be the configuration occupied by $\mathcal{B}$ at some prescribed time.]

Since B_r and B_t are configurations of $\mathcal{B}$ there exists a bijection mapping $\boldsymbol{\chi} : \mathrm{B}_r \to \mathrm{B}_t$ such that

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\chi}(\mathbf{X}) \qquad \text{for all } \mathbf{X} \in \mathrm{B}_r, \\ \mathbf{X} &= \boldsymbol{\chi}^{-1}(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \mathrm{B}_t. \end{aligned} \tag{3.1}$$

The mapping $\boldsymbol{\chi}$ is called the *deformation* of the body *from* B_r *to* B_t.

Since B_t depends on t we write

$$\mathbf{x} = \boldsymbol{\chi}_t(\mathbf{X}), \quad \mathbf{X} = \boldsymbol{\chi}_t^{-1}(\mathbf{x}) \tag{3.2}$$