If you look at the loop of any modern rollercoaster, such as the ones shown in figure 1, you'll see that they are not circular, but in a sort of inverted tear-drop shape.

Figure 1: Some examples of rollercoaster loops

Early rollercoaster loops were circular, but there are problems with circular loops. To start with, in order to ensure that the carriages make it round the loop, either the loop has to be small and tight, or the carriage has to enter the loop at a very high speed. This means that there will be sudden change from a straight track to the curved loop, and the resulting acceleration is both uncomfortable and dangerous. Passengers riding such rollercoasters used to break their collarbones with the forces that snapped their heads towards their chests.

The trick to designing a scary—but safe!—rollercoaster loop is to ensure that there is only a gradual acceleration between straight and curved track.

A curve that moves between a straight line and another curve (such as a circle), is called a *transition curve*. Such curves are used in rollercoasters, and on roads, for example as exit ramps from freeways. They are also used in railway design.

One particular curve which is much used is the *Euler spiral*, named for the Swiss mathematician Leonhard Euler (1720–1723) who first analysed it. It is also called the called the *Cornu spiral* after the French physicist Marie Alfred Cornu (1841–1902) who discussed the curve as part of his researches into light waves. It can be defined by the parametric equations

$$x(t) = \int_0^t \cos\left(\frac{\pi}{2}s^2\right)\, ds, \qquad y(t) = \int_0^t \sin\left(\frac{\pi}{2}s^2\right)\, ds$$

The integrals in these equations are called *Fresnel integrals* after the French engineer and physicist Augustin-Jean Fresnel (1788–1827).

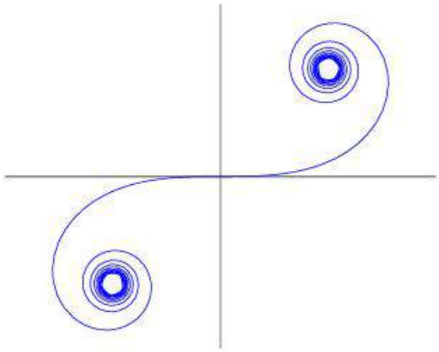

Figure 2: A clothoid curve

The curve is shown in figure 2. Note that the further it gets away from the origin, the tighter it curves in on itself.

Beacuse the curve looks like thread being wound around a couple of sticks, it is also called a *clothoid curve*, after Clotho, the Fate in ancient Greek mythology, who was said to have "spun the thread of human life".

So one way to design a rollercoaster loop using this curve is to follow the curve around from the origin until it is flat. And we can find this analytically because

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin\left(\frac{\pi}{2}t^2\right)}{\cos\left(\frac{\pi}{2}t^2\right)} = \tan\left(\frac{\pi}{2}t^2\right).$$

So if $\tan(\pi/2\,t^2) = 0$ for some non-zero value, we might try $(\pi/2)t^2 = \pi$ and so $t = \sqrt{2}$. Then we can plot the curve between $t = 0$ and $t = \sqrt{2}$, and follow up with its mirror image, to make a lovely rollercoaster loop. This process is shown in figure 3.

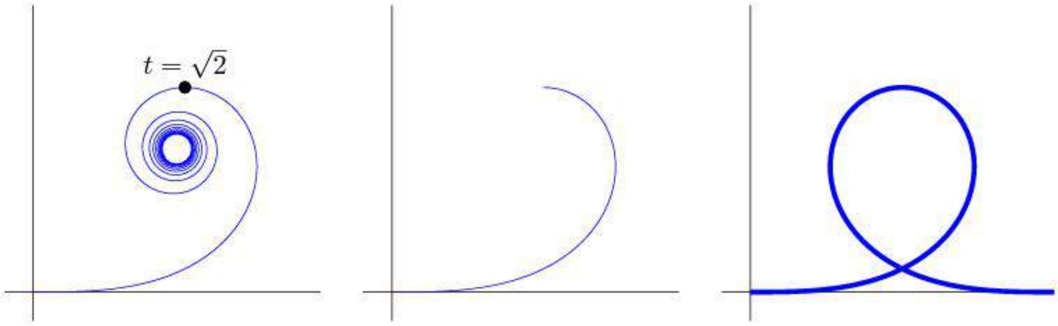

Figure 3: Designing a loop

Problem 1. Find a way to plot a clothoid curve, using any computational or mathematical tools you need.

Problem 2. Find an example where such curves are used in road or railway design.