1202 2012-2013 Exercise 4

1 (i) Let $C_n = \{e, x, ..., x^{n-1}\}$ ($x^n = e$) be the cyclic cyclic group of order n and let H be a subgroup of C_n. By considering $min\{i > 0 : x^i \in H\}$ show that $H = \langle x^m \rangle$ for some $m|n$.

***(ii)** Find all subgroups of C_{18}, explaining your answer.

2(i) Find (a) 4^{1602} (mod 17), (b) 4^{-1} (mod 17), (c) 4^{1597} (mod 17)

***(ii)** Find (a) 11^{-1} (mod 13), (b) 11^{1198} (mod 13)

3 Let G be a group in which $g^2 = e$ for all $g \in G$. Show that G is abelian.

***4** Thoughout this question, let G be a finite group and H and K subgroups of G.

(i) Prove that $H \cap K$ is a subgroup of G.

(ii) Let HK denote the set $\{hk : h \in H, k \in K\}$. Prove that if $H \cap K = \{e\}$ then $|HK| = |H||K|$.

(iii) Prove that if G is abelian then HK is a subgroup of G.

(iv) Prove that if $|G| = 150$, $|H| = 6$, $|K| = 25$ then $G = HK$.

***5** Let G and H be groups. A map $\phi : G \longrightarrow H$ is said to be an *isomorphism* if ϕ is bijective and $\phi(g_1g_2) = \phi(g_1)\phi(g_2)$ for all $g_1, g_2 \in G$. If there exists an isomorphism $\phi : G \longrightarrow H$ then G and H are said to be *isomorphic* and we write $G \cong H$. You can think of isomorphic groups as being the same group with different names: if we write down the group table for G and rename elements we get the group table for H.

(i) Show that if ϕ is as above then $\phi(e_G) = e_H$ and $\phi(g^{-1}) = (\phi(g))^{-1}$ for all $g \in G$. (Here e_G is identity element of G and e_H the identity element of H.)

(ii) Write down the group tables for C_4 and for $\mathbf{Z}_5^*$. Find an isomorphism $\phi : C_4 \longrightarrow \mathbf{Z}_5^*$.

(iii) Recall that if X and Y are sets then the Cartesian product is the set of ordered pairs $X \times Y = \{(x, y) : x \in X, y \in Y\}$. Show that if G and H are groups then $G \times H$ is a group under the group operation $(g_1, h_1) \star (g_2, h_2) = (g_1g_2, h_1h_2)$.

(iv) Write down the group table for $C_2 \times C_2$. Show that $C_2 \times C_2$ is not isomorphic to C_4.

The following questions are optional and harder

6 Let G be a group of order 4. By first considering the order of elements in G, show that either $G \cong C_4$ or $G \cong C_2 \times C_2$. [Thus, up to isomorphism, there are just two groups of order 4.]

7 Let G be a group of order 6. By first considering the order of elements in G, show that either $G \cong C_6$ or $G \cong D_3$, where D_3 is the symmetry group of the triangle, $D_3 = \langle x, y : x^3 = e, y^2 = e, yx = x^2y \rangle = \{e, x, x^2, y, xy, xy^2\}$. [Thus, up to isomorphism, there are just two groups of order 6.]

Please attempt Questions 1 - 5. Please hand in the **assessed** *questions (the questions marked with a *) on Monday 25 February at the lecture. There will be a problem class on Wednesday 20 February to help with this sheet.*