1. The general form of a *Lienard* system is

$$\frac{dx}{dt} = y - f(x)$$
$$\frac{dy}{dt} = -x$$

If $f(x) = \mu(x^3 - x)$, this is the system form of *Van der Pol's equation.* Find all the equilibria of the Van der Pol equation and characterize them as sources, sinks, etc. Make sure to analyze the influence of μ on the characterization.

For more information on this application:

Frequency demultiplication, Van der Pol, B. and van der Mark, Nature, 120, 363-364, (1927).

A Lienard Oscillator Resonant Tunnelling Diode-Laser Diode Hybrid Integrated Circuit: Model and Experiment,, Slight, T.J.; Romeira, B.; Liquan Wang; Figueiredo, J.M.L.; Wasige, E.; Ironside, C.N. .; IEEE J. Quantum Electronics, Volume 44, Issue 12, Dec. 2008 Page(s):1158 – 1163.

2. Komarova and co-workers[*] have devised a simple 2-D dynamical system model of cell dynamics in bone remodeling. The equations associated with their model are:

$$\frac{dx_1}{dt} = \alpha_1 x_1^{g_{11}} x_2^{g_{21}} - \beta_1 x_1$$
$$\frac{dx_2}{dt} = \alpha_2 x_1^{g_{12}} x_2^{g_{22}} - \beta_2 x_2$$

Here, we're interested only in positive values of x_1 and x_2, $\beta_1 > 0$ and $\beta_2 > 0$, and the parameters g_{ij} may take on any real values.

Find all equilibria (again, we're only interested in the first quadrant) of the system. Identify degenerate cases. Find values of the parameters for which at least one of the equilibria is a source, other values for which at least one is a sink, others for which at least one is a saddle, others that yield a center, others that yield a stable focus, and others that yield an unstable focus.

Sketch a phase diagram corresponding to each of these cases.

Try to make some general statements about the structure of equilibrium as a function of the parameters.

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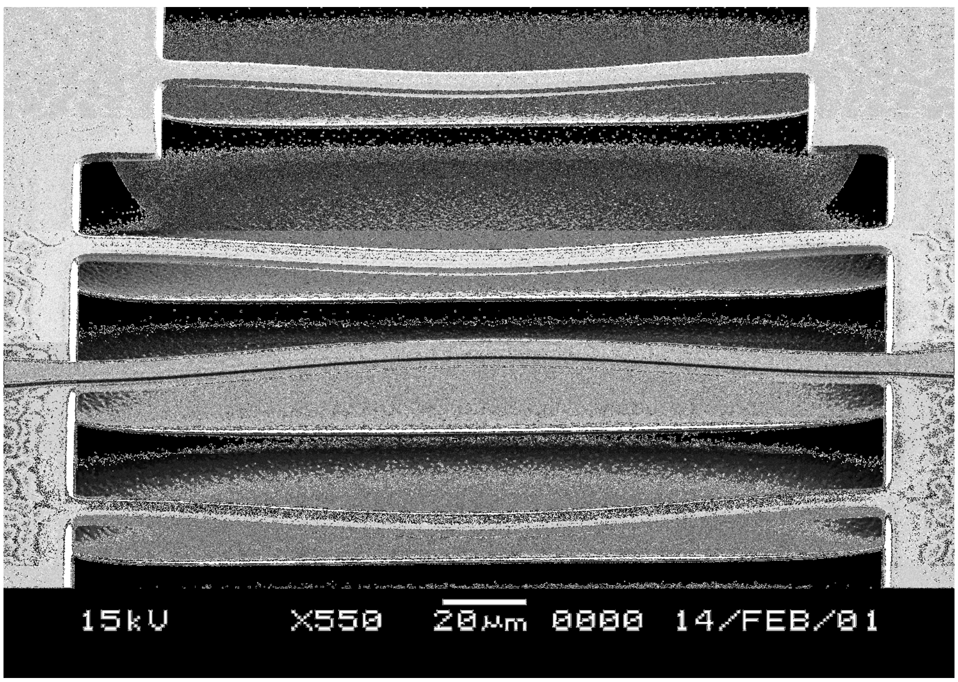
[*] *Mathematical model predicts a critical role for osteoclast autocrine regulation in the control of bone remodeling*, S. V. Komarova, R. J. Smith, S. J. Dixon, S. M. Sims, L. M. Wahl, Bone 2003;33:206-215.

3. The planar dynamical system

$$x' = y$$
$$y' = \frac{1}{2}x^3 - x$$

arises in the analysis of the equilibrium shape of a doubly-clamped bimetallic beam. An example of such a beam is shown in the scanning electron micrograph below. These equations have been scaled, but in essence x is the slope of the beam and y is the derivative of the slope—essentially the curvature of the beam since the slope is small—and the independent variable is position along the beam.



Find all of the equilibria of this system. Linearize the system at each of the equilibria and characterize each equilibrium as a source, a sink, a center, or whatever. Sketch out the phase diagram.

For more information on this application:

Temperature-Dependent Vibrations of Bilayer MicroBeams, D. S. Ross, A. Cabal, D. Trauernicht, and J. Lebens, Sensors and Actuators, Vol. 119, #2, 537-542 2004.

Snap-Through BiLayer Microbeam, A. Cabal and D. S. Ross, Proceedings, 5th International Microsystems Conference, Computational Publicasion, San Juan 2002.

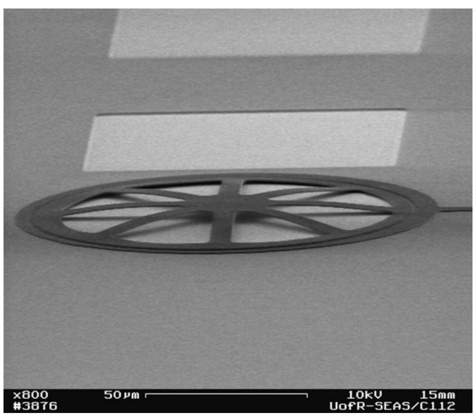
U.S. Patent # 6,953,240: Snap-Through Thermal Actuator.

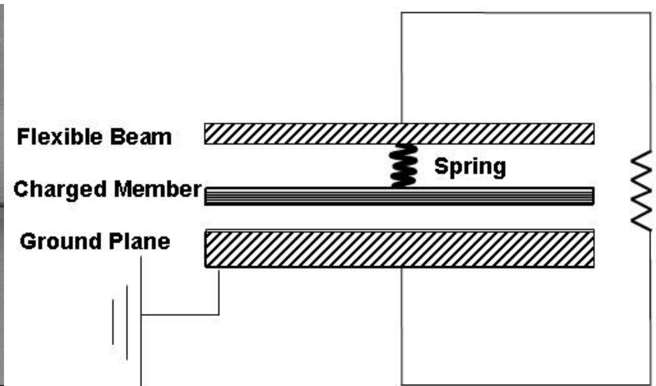
U.S. Patent # 6,572,220: Beam Micro-Actuator with Tunable or Stable Amplitude Suited for Ink Jet Printing.

4. The 3-dimensional dynamical system

$$x' = y$$
$$\rho h y' = -kx + \frac{q^2}{2\varepsilon_0}$$
$$RAq' = -\frac{d_1}{\varepsilon}(q + q_0) - \frac{d_2 - x}{\varepsilon_0} q$$

is the mathematical expression of a simple model of an embedded-charge MEMS energy harvester. A schematic of the device is shown below and to the right, and an SEM of a prototype is shown below to the left. Here, $x(t)$, $y(t)$, and $q(t)$ are the dependent variables, time t is the independent variable, and everything else is a positive constant.





Here, consider the simplified system

$$x' = y$$
$$y' = -x + q^2$$
$$q' = -(q + q_0) - (1 - x)q$$

Note that there is one parameter remaining, q_0. Find the equilibria of the system, linearize at each equilibrium, and characterize the stability of the equilibria. Bear in mind the dependence on q_0.

For more information on this application:

Design and Modeling of a Micro-Energy Harvester Using an Embedded Charge Layer, S. Gracewski, P. Funkenbusch, D. S. Ross, J. Zia, M. Potter , J. Micromech. Microeng. Vol. 16, 235-241, 2006.

Dynamics of Embedded Charge MEMS Energy Harvesting,, C. Lutzer, D. S. Ross, *The Dynamics of Embedded-Charge Microenergy Harvesting*, C. Lutzer, D. S. Ross, J. Comp. Nonlin. Dyn, Vol.5, #2, 2010.

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5. The planar dynamical system,

$$x' = x(\varepsilon_x(1-x) - \varepsilon_y y)$$
$$y' = y(\varepsilon_y(1-y) - \varepsilon_x x)$$

is a version of the *competing species* system. Here, ε_x and ε_y are constants.

Find all equilibria of this system.

This system recently arose in research that professors Thurston and Wahle and I are doing into the influence of mixture free energies on the scattering of light from those mixtures. (That is, it has nothing at all to do with competing species except that the equations are the same.) In that application, x and y are normalized densities so they're positive and $x + y \le 1$. So, from here on, restrict your attention to the triangle $x \ge 0 \quad y \ge 0 \quad x + y \le 1$.

Show that the vertices of this triangle are equilibria for any choice of ε_x and ε_y. Show that except in certain degenerate cases, one vertex is a source, one is a sink, and one is a saddle. Show that there are no equilibria in the interior of the triangle. Characterize the degenerate cases in terms of ε_x and ε_y. Show that trajectories that begin in the triangle remain in the triangle.

For more information on this application:

Differential Equations, Dynamical Systems, and an Introduction to Chaos, Hirsch, Smale, & Devany, Elsevier, 2002.

Notes on the Competing Species Equations, by Professor Wahle.

6. Lemaire and co-workers[*],developed a model of cell signaling in bone remodeling that is different from that of Komorova, which was the subject of problem 2. The model of Lemaire is a *mechanistic* model, that is, it is derived from facts about the underlying mechanisms of the phenomenon. Their system is

$$\frac{dR}{dt} = D_R \pi_C - \frac{D_B}{\pi_C} R$$

$$\frac{dB}{dt} = \frac{D_B}{\pi_C} R - k_B B$$

$$\frac{dC}{dt} = \frac{k_1 B}{1 + k_2 R} - D_A \pi_C C$$

Here,

$$\pi_C(C) = \frac{C + f\, C_0}{C + C_0}$$

and everything except $R(t)$, $B(t),$ and $C(t)$ is a positive constant. The functions $R(t)$, $B(t),$ and $C(t)$ are cell concentrations, the concentrations of *responding osteoblasts, active osteoblasts,* and *osteoclasts*, respectively. Concentrations are always non-negative, of course, so we are interested only in non-negative values of $R(t)$, $B(t),$ and $C(t)$. See Lemaire's paper for the details of the derivation of the model, it's very interesting. Note that the function $\pi_C(C) = \frac{C + f\, C_0}{C + C_0}$ is a rational function of the osteoclast concentration; it has nothing to do with the ratio of the circumference of a circle to its diameter.

Consider the particular case

$$\frac{dR}{dt} = \pi_C - \frac{R}{\pi_C}$$

$$\frac{dB}{dt} = \frac{R}{\pi_C} - B$$

$$\frac{dC}{dt} = \frac{\frac{5}{4} B}{1 + R} - \pi_C C$$

with,

$$\pi_C(C) = \frac{C + \frac{1}{2}}{C + 1}$$

[*] Vincent Lemairea,, Frank L. Tobina, Larry D. Grellera, Carolyn R. Choa, Larry J. Suvab, *Modeling the interactions between osteoblast and osteoclast activities in bone remodeling,* Journal of Theoretical Biology 229 (2004) 293–309.

Find all realistic equilibria of this system, that is, all equilibria with non-negative values of the concentrations. Prove that you have identified all of them. Characterize the stability of the equilibria; that is, prove whether each is a source or a sink or something else. If there are any centers, identify the period of oscillation associated with each . If there are any sinks, identify the relaxation times of those sinks, that is, identify how long it takes for a solution that is slightly perturbed from the equilibrium to settle back to the equilibrium. (Mathematically, we know that such perturbed solutions approach the equilibrium asymptotically, they never actually arrive at the equilibrium. This is not what I'm looking for, I'm looking for the time it takes, roughly, for the perturbation to decrease by half, or by a factor of 10.)

7. Consider the following dynamical system, which is a discretized version of a reaction-diffusion system:

$$\frac{dF_1}{dt} = -2F_1 + F_2 + N^2 cF_1(1 - F_1)$$

$$\frac{dF_j}{dt} = F_{j-1} - 2F_j + F_{j+1} + N^2 cF_j(1 - F_j)$$

$$\frac{dF_N}{dt} = F_{N-1} - 2F_N + N^2 cF_{N-1}(1 - F_{N-1})$$

Here, N is an integer that is greater than 2 and c is a real constant. This term in the equations,

$$F_{j-1} - 2F_j + F_{j+1} \ ,$$

represents the effects of diffusion, and this term,

$$N^2 cF_j(1 - F_j),$$

represents the effect of a reaction.

The point $F_j = 0, j = 1, \ldots N$ is an equilibrium for any N and any c. Show that for any fixed N there is a positive number η such that the equilibrium is stable for all values of c less than η and is unstable for all values of c greater than . Estimate η, and find the limit of η as N approaches ∞ exactly.

In working on this problem, it may be useful for you to know that if $\theta = \frac{j\pi}{N}$, $1 \le j \le N$, the vector

$$(\sin(\theta)\,, \sin(2\theta)\,, \sin(3\theta)\,, \ldots, \sin(N\theta))$$

is an eigenvector of the tridiagonal matrix

$$\begin{bmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & & & & \\ & & & \ddots & & & \\ & & & & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{bmatrix}$$

(If you use this fact, of course, prove it.) The value of η at which the behavior of the system changes is called a *bifurcation point* of the parameter c, and the change is a *bifurcation.* Try to devise an informal explanation of what is happening in the system when the bifurcation occurs. If you're interested in an application of this system to biology (which may give you an idea about the meaning of the bifurcation), take a look at the classic paper by Fisher[*].

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[*] R.A. Fisher, *The Wave of Advance of Advantageous Genes,* Annals of Eugenics, v. 7: 355-369 (1937)

8. The system

$$z' = z(1-\xi)$$
$$\xi' = (h-1)\xi^2 + (1-c)\xi + \frac{c}{K}z$$

arises in the analysis of a population demographics model[*] of Easter Island that we will consider in greater detail later in the quarter. The parameters h, c, and K are all positive. Characterize the stability of the equilibrium at the origin for all values of the parameters. For any cases in which the equilibrium is stable, identify the relaxation time associated with the equilibrium (see problem 6). For any cases in which the equilibrium is a center, or a spiral sink, find the period of oscillation associated with the equilibrium.

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[*] *Booming and Crashing Populations and Easter Island*, Basener, B. and Ross, D. S., SIAM J. Appl. Math., Vol. 65, #2, 684-701, (2005).