COMP 2804 — Assignment 4

Due: April 3, before 23:59 pm, in the course drop box in Herzberg 3115.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.
- When presenting proofs, every step should be justified.
- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

> If you submit your solutions using LaTeX, you will get one bonus mark.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: The Ontario Lottery and Gaming Corporation (OLG) offers the following lottery game:

- OLG chooses a winning number w in $S = \{0, 1, 2, \ldots, 999\}$.
- If John wants to play, he pays \$1 and chooses a number x in S.
  - If $x = w$, then John receives \$700 from OLG. In this case, John wins \$699.
  - Otherwise, $x \neq w$ and John does not receive anything. In this case, John loses \$1.

Assume that

- John plays this game once per day for one year (i.e., for 365 days),
- each day, OLG chooses a new winning number,
- each day, John chooses x uniformly at random from the set S, independently from previous choices.

Define the random variable X to be the total amount of dollars that John wins during one year. Determine the expected value $\mathbb{E}(X)$. (*Hint:* Use Linearity of Expectation.)

Question 3: The Ottawa Senators and the Toronto Maple Leafs play a best-of-seven series: These two hockey teams play against each other until one of them has won four games. Assume that

- in any game, the Sens have a probability of 3/4 of defeating the Leafs,
- the results of the games are independent.

Determine the probability that seven games are played in this series.

Question 4: Let $X_1, X_2, \ldots, X_n$ be a sequence of mutually independent random variables. For each i with $1 \le i \le n$,

- the variable X_i is equal to either 0 or $n+1$,
- $\mathbb{E}(X_i) = 1$.

Determine

$$\Pr(X_1 + X_2 + \cdots + X_n \le n).$$

Question 5: Consider the set $V = \{1, 2, \ldots, n\}$ and let p be a real number with $0 < p < 1$. We construct a graph $G = (V, E)$ with vertex set V, whose edge set E is determined by the following random process: Each unordered pair $\{i, j\}$ of vertices, where $i \neq j$, occurs as an edge in E with probability p, independently of the other unordered pairs.

A *triangle* in G is an unordered triple $\{i, j, k\}$ of distinct vertices, such that $\{i, j\}$, $\{j, k\}$, and $\{k, i\}$ are edges in G.

Define the random variable X to be the total number of triangles in the graph G. Determine the expected value $\mathbb{E}(X)$. (*Hint:* Use indicator random variables.)

Question 6: Michiel's Craft Beer Company (MCBC) sells n different brands of India Pale Ale (IPA). When you place an order, MCBC sends you one bottle of IPA, chosen uniformly at random from the n different brands, independently of previous orders.

Simon Pratt wants to try all different brands of IPA. He repeatedly places orders at MCBC (one bottle per order) until he has received at least one bottle of each brand.

Define the random variable X to be the total number of orders that Simon places. Determine the expected value $\mathbb{E}(X)$. (*Hint:* Use Linearity of Expectation. If Simon has received exactly i different brands of IPA, how many orders does he expect to place until he receives a new brand?)

Question 7: MCBC still sells n different brands of IPA. As in the previous question, when you place an order, MCBC sends you one bottle of IPA, chosen uniformly at random from the n different brands, independently of previous orders.

Simon Pratt places m orders at MCBC. Define the random variable X to be the total number of distinct brands that Simon receives. Determine the expected value $\mathbb{E}(X)$. (*Hint:* Use indicator random variables.)

Question 8: You are given an array $A[0 \ldots n-1]$ of n numbers. Let d be the number of *distinct* numbers that occur in this array. For each i with $0 \leq i \leq n-1$, let N_i be the number of elements in the array that are equal to $A[i]$.

- Show that $d = \sum_{i=0}^{n-1} 1/N_i$.

Consider the following algorithm:

Step 1: Choose an integer k in $\{0, 1, 2, \ldots, n-1\}$ uniformly at random, and let $a = A[k]$.

Step 2: Traverse the array and compute the number N_k of times that a occurs.

Step 3: Return the value $X = n/N_k$.

- Determine the expected value $\mathbb{E}(X)$ of the random variable X. (*Hint:* Use the definition of expected value, i.e., Definition 6.4.1 in the notes.)