Math 422/522 - Homework 7

1) The *Laguerre polynomial* of degree n is defined by the equation

$$L_n(t) = e^t \frac{d^n}{dt^n}(t^n e^{-t}).$$

Prove that

$$\mathcal{L}\{L_n(t)\} = \frac{n!}{s}\left(\frac{s-1}{s}\right)^n.$$

2) Let $f(t) = F(t)$ when $0 < t < a$, and let $f(t)$ be *periodic*, of period a, so that $f(t+a) = f(t)$. By writing

$$\mathcal{L}\{f(t)\} = \int_0^a e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt + \int_{2a}^{3a} e^{-st} f(t)\,dt + \cdots$$

and transforming each integral in such a way that in each case the range of integration is $(0, a)$, show that

$$\mathcal{L}\{f(t)\} = \int_0^a e^{-st} F(t)\,dt\,[1 + e^{-as} + e^{-2as} + \cdots],$$

and hence, when $s > 0$,

$$\mathcal{L}\{f(t)\} = \frac{\int_0^a e^{-st} F(t)\,dt}{1 - e^{-as}}.$$

3) Apply the result of Problem 2 to the "square-wave function" for which $F(t) = 1$ when $0 < t < a/2$ and $F(t) = -1$ when $a/2 < t < a$. Show that the transform of this function is

$$\frac{(1 - e^{-as/2})^2}{s(1 - e^{-as})} = \frac{1}{s}\frac{1 - e^{-as/2}}{1 + e^{-as/2}} = \frac{1}{s}\tanh\frac{as}{4}.$$

4) If $f(t)$ is the "staircase function" such that $f(t) = b$ when $0 < x < a$, $f(t) = 2b$ when $a < x < 2a$, and so forth, show that

$$\mathcal{L}\{f(t)\} = \frac{b}{s(1 - e^{-as})}.$$