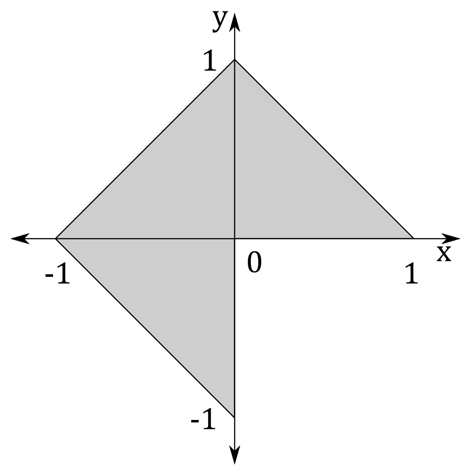
1. The random variables X, Y, and Z are independent and have a normal distribution. Specifically, $X \sim \mathsf{N}(0,1)$, $Y \sim \mathsf{N}(1,4)$, and $Z \sim \mathsf{N}(-1,2)$.

   (a) Determine the expectation of $U = 2X + 3Z$ and the distribution of $V = X + Y - 2Z$. [1]

   (b) Compute the covariance of U and V. [1]

2. We draw a random vector (X, Y) uniformly from the diamond $(-1,0)$–$(0,1)$–$(1,0)$–$(0,-1)$ *without* its lower-right corner (see figure).



   (a) Write down the joint pdf of X and Y, clearly specifying where it is zero. [1]

   (b) Determine the marginal pdf of Y. [1]

3. Random variables X and Y have a joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} x \cos y & 0 < x < \pi/2, \quad 0 < y < x\,, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Determine the conditional probability density function of Y given $X = x$. [1]

   (b) Evaluate $\mathbb{P}(X + Y \geqslant \pi/2)$. [1]

4. Let X, Y be independent $\mathsf{Ber}(p)$ distributed random variables. Let $U = X + Y$ and $V = |X - Y|$.

   (a) Give a table for the joint pmf of U and V. [1]

   (b) Calculate the *correlation* $\varrho(U,V) = \mathrm{Cov}(U,V)/\sqrt{\mathrm{Var}(U)\mathrm{Var}(V)}$. Are U and V independent when $p = 1/2$? [1]

5. The number of defective transistors on an integrated circuit-board has a Poisson distribution, with parameter 100, and the number of defective resistors on the same circuit-board has a Poisson distribution with parameter 1.

   Let X be the number of defective transistors, and Y be the number of defective resistors. Assume that X and Y are *independent*.

(a) Show that the total number $Z = X + Y$ of defective components (either transistors or resistors) has a Poisson distribution with parameter 101. [1]

(b) Given that 101 components have failed, what is the probability that only 1 of them is a transistor? [1]

6. Suppose that a widget manufacturing plant has two production systems working in parallel to produce widgets. The first widget system is older, and produces 10000 widgets a day, with each widget being defective with probability 0.005, independently. The second widget system is brand new, and produces 25000 widgets a day, with each widget being defective with probability 0.002, independently. Moreover, the systems operate independently. At the end of each day, all of the day's widgets are collected together in a box.

   Let X and Y be the number of widgets the first and second system produce per day, respectively.

   (a) What are the expectations and variances of X and Y? [1]

   (b) Using the central limit theorem and the table of the standard normal cdf, approximate the probability that more than 110 defective widgets are made in a day. [1]

Total [12]

Formula Sheet

Standard normal distribution

This table gives the cumulative distribution function (cdf) Φ of a $\mathsf{N}(0,1)$-distributed random variable Z.

$$\Phi(z) = \mathbb{P}(Z \leqslant z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx.$$

The last column gives the probability density function (pdf) φ of the $\mathsf{N}(0,1)$-distribution

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	$\varphi(z)$
0.0	5000	5040	5080	5120	5160	5199	5239	5279	5319	5359	0.3989
0.1	5398	5438	5478	5517	5557	5596	5636	5675	5714	5753	0.3970
0.2	5793	5832	5871	5910	5948	5987	6026	6064	6103	6141	0.3910
0.3	6179	6217	6255	6293	6331	6368	6406	6443	6480	6517	0.3814
0.4	6554	6591	6628	6664	6700	6736	6772	6808	6844	6879	0.3683
0.5	6915	6950	6985	7019	7054	7088	7123	7157	7190	7224	0.3521
0.6	7257	7291	7324	7357	7389	7422	7454	7486	7517	7549	0.3332
0.7	7580	7611	7642	7673	7704	7734	7764	7794	7823	7852	0.3123
0.8	7881	7910	7939	7967	7995	8023	8051	8078	8106	8133	0.2897
0.9	8159	8186	8212	8238	8264	8289	8315	8340	8365	8389	0.2661
1.0	8413	8438	8461	8485	8508	8531	8554	8577	8599	8621	0.2420
1.1	8643	8665	8686	8708	8729	8749	8770	8790	8810	8830	0.2179
1.2	8849	8869	8888	8907	8925	8944	8962	8980	8997	9015	0.1942
1.3	9032	9049	9066	9082	9099	9115	9131	9147	9162	9177	0.1714
1.4	9192	9207	9222	9236	9251	9265	9279	9292	9306	9319	0.1497
1.5	9332	9345	9357	9370	9382	9394	9406	9418	9429	9441	0.1295
1.6	9452	9463	9474	9484	9495	9505	9515	9525	9535	9545	0.1109
1.7	9554	9564	9573	9582	9591	9599	9608	9616	9625	9633	0.0940
1.8	9641	9649	9656	9664	9671	9678	9686	9693	9699	9706	0.0790
1.9	9713	9719	9726	9732	9738	9744	9750	9756	9761	9767	0.0656
2.0	9772	9778	9783	9788	9793	9798	9803	9808	9812	9817	0.0540
2.1	9821	9826	9830	9834	9838	9842	9846	9850	9854	9857	0.0440
2.2	9861	9864	9868	9871	9875	9878	9881	9884	9887	9890	0.0355
2.3	9893	9896	9898	9901	9904	9906	9909	9911	9913	9916	0.0283
2.4	9918	9920	9922	9925	9927	9929	9931	9932	9934	9936	0.0224
2.5	9938	9940	9941	9943	9945	9946	9948	9949	9951	9952	0.0175
2.6	9953	9955	9956	9957	9959	9960	9961	9962	9963	9964	0.0136
2.7	9965	9966	9967	9968	9969	9970	9971	9972	9973	9974	0.0104
2.8	9974	9975	9976	9977	9977	9978	9979	9979	9980	9981	0.0079
2.9	9981	9982	9982	9983	9984	9984	9985	9985	9986	9986	0.0060
3.0	9987	9987	9987	9988	9988	9989	9989	9989	9990	9990	0.0044
3.1	9990	9991	9991	9991	9992	9992	9992	9992	9993	9993	0.0033
3.2	9993	9993	9994	9994	9994	9994	9994	9995	9995	9995	0.0024
3.3	9995	9995	9995	9996	9996	9996	9996	9996	9996	9997	0.0017
3.4	9997	9997	9997	9997	9997	9997	9997	9997	9997	9998	0.0012
3.5	9998	9998	9998	9998	9998	9998	9998	9998	9998	9998	0.0009
3.6	9998	9998	9999	9999	9999	9999	9999	9999	9999	9999	0.0006

Example: $\Phi(1.65) = \mathbb{P}(Z \leqslant 1.65) = 0.9505$

Summary of Formulas

1. **Sum rule:** $\boxed{\mathbb{P}(\bigcup_i A_i) = \sum_i \mathbb{P}(A_i)}$, when $A_1, A_2, \ldots$ are disjoint.

2. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

3. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

4. **Cdf** of X: $F(x) = \mathbb{P}(X \leqslant x)$, $x \in \mathbb{R}$.

5. **Pmf** of X: (discrete r.v.) $f(x) = \mathbb{P}(X = x)$.

6. **Pdf** of X: (continuous r.v.) $f(x) = F'(x)$.

7. For a discrete r.v. X: $\boxed{\mathbb{P}(X \in B) = \sum_{x \in B} \mathbb{P}(X = x).}$

8. For a continuous r.v. X with pdf f:
$\boxed{\mathbb{P}(X \in B) = \int_B f(x)\, dx.}$

9. In particular (continuous), $F(x) = \int_{-\infty}^{x} f(u)\, du$.

10. Similar results 7-8 hold for random vectors, e.g. $\mathbb{P}((X,Y) \in B) = \iint_B f_{X,Y}(x,y)\, dx\, dy$.

11. Marginal from joint pdf: $f_X(x) = \int f_{X,Y}(x,y)\, dy$.

12. **Important discrete distributions**:

Distr.	pmf	$x \in$
$\mathsf{Ber}(p)$	$p^x(1-p)^{1-x}$	$\{0,1\}$
$\mathsf{Bin}(n,p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	$\{0,1,\ldots,n\}$
$\mathsf{Poi}(\lambda)$	$\mathrm{e}^{-\lambda} \frac{\lambda^x}{x!}$	$\{0,1,\ldots\}$
$\mathsf{Geom}(p)$	$p(1-p)^{x-1}$	$\{1,2,\ldots\}$

13. **Important continuous distributions**:

Distr.	pdf	$x \in$
$\mathsf{U}[a,b]$	$\frac{1}{b-a}$	$[a,b]$
$\mathsf{Exp}(\lambda)$	$\lambda\, \mathrm{e}^{-\lambda x}$	$\mathbb{R}_+$
$\mathsf{Gam}(\alpha,\lambda)$	$\frac{\lambda^\alpha x^{\alpha-1} \mathrm{e}^{-\lambda x}}{\Gamma(\alpha)}$	$\mathbb{R}_+$
$\mathsf{N}(\mu,\sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$\mathbb{R}$

14. **Conditional probability:** $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$.

15. **Law of total probability**: $\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\, \mathbb{P}(B_i)$, with $B_1, B_2, \ldots, B_n$ a partition of Ω.

16. **Bayes' Rule:** $\mathbb{P}(B_j | A) = \frac{\mathbb{P}(B_j)\, \mathbb{P}(A|B_j)}{\sum_{i=1}^{n} \mathbb{P}(B_i)\, \mathbb{P}(A|B_i)}$.

17. **Product rule**: $\boxed{\mathbb{P}(A_1 \cdots A_n) = \mathbb{P}(A_1)\, \mathbb{P}(A_2 \mid A_1) \cdots \mathbb{P}(A_n \mid A_1 \cdots A_{n-1}).}$

18. **Memoryless property** (Exp and Geom distribution): $\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t)$, $\forall s, t$.

19. **Independent events**: $\mathbb{P}(A \cap B) = \mathbb{P}(A)\, \mathbb{P}(B)$.

20. **Independent r.v.'s:** (discrete) $\boxed{\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) = \prod_{k=1}^{n} \mathbb{P}(X_k = x_k)\ .}$

21. **Independent r.v.'s:** (continuous) $\boxed{f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \prod_{k=1}^{n} f_{X_k}(x_k)\ .}$

22. **Expectation** (discr.): $\mathbb{E}X = \sum_x x\, \mathbb{P}(X = x)$.

23. (of function) $\mathbb{E}\, g(X) = \sum_x g(x)\, \mathbb{P}(X = x)$.

24. **Expectation** (cont.): $\mathbb{E}X = \int x\, f(x)\, dx$.

25. (of function) $\mathbb{E}\, g(X) = \int g(x) f(x)\, dx$,

26. Similar results 18–21 hold for random vectors.

27. **Expected sum :** $\mathbb{E}(aX + bY) = a\, \mathbb{E}X + b\, \mathbb{E}Y$.

28. **Expected product** (only if X, Y independent): $\mathbb{E}[X\, Y] = \mathbb{E}X\, \mathbb{E}Y$.

29. **Markov inequality:** $\mathbb{P}(X \geqslant x) \leqslant \frac{\mathbb{E}X}{x}$.

30. $\mathbb{E}X$ **and Var**(X) **for various distributions**:

	$\mathbb{E}X$	$\mathrm{Var}(X)$
$\mathsf{Ber}(p)$	p	$p(1-p)$
$\mathsf{Bin}(n,p)$	np	$np(1-p)$
$\mathsf{Geom}(p)$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
$\mathsf{Poi}(\lambda)$	λ	λ
$\mathsf{U}(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$\mathsf{Exp}(\lambda)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
$\mathsf{Gam}(\alpha,\lambda)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$
$\mathsf{N}(\mu,\sigma^2)$	μ	σ^2

31. n**-th moment:** $\mathbb{E}X^n$.

32. **Covariance:** $\mathrm{cov}(X,Y) = \mathbb{E}(X - \mathbb{E}X)(Y - \mathbb{E}Y)$.

33. **Properties of Var and Cov**:

$\mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$.
$\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$.
$\mathrm{cov}(X,Y) = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y$.
$\mathrm{cov}(X,Y) = \mathrm{cov}(Y,X)$.
$\mathrm{cov}(aX + bY, Z) = a\, \mathrm{cov}(X,Z) + b\, \mathrm{cov}(Y,Z)$.
$\mathrm{cov}(X,X) = \mathrm{Var}(X)$.
$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\, \mathrm{cov}(X,Y)$.
X and Y independent $\Longrightarrow \mathrm{cov}(X,Y) = 0$.

34. **Probability Generating Function (PGF)**: $G(z) := \mathbb{E}\, z^N = \sum_{n=0}^{\infty} \mathbb{P}(N = n) z^n, \qquad |z| < 1$.

35. **PGFs for various distributions:**

$\mathsf{Ber}(p)$	$1 - p + zp$
$\mathsf{Bin}(n,p)$	$(1 - p + zp)^n$
$\mathsf{Geom}(p)$	$\frac{z\, p}{1 - z\,(1-p)}$
$\mathsf{Poi}(\lambda)$	$\mathrm{e}^{-\lambda(1-z)}$

36. $\mathbb{P}(N = n) = \frac{1}{n!}\, G^{(n)}(0)$. (n-th derivative, at 0)

37. $\mathbb{E}N = G'(1)$

38. $\mathrm{Var}(N) = G''(1) + G'(1) - (G'(1))^2$.

39. **Moment Generating Function (MGF)**: $M(s) = \mathbb{E}\, \mathrm{e}^{sX} = \int_{-\infty}^{\infty} \mathrm{e}^{sx} f(x)\, dx$, $s \in I \subset \mathbb{R}$, for r.v.'s X for which all moments exist.

40. **MGFs for various distributions:**

$\mathsf{U}(a,b)$	$\frac{e^{bs}-e^{as}}{s(b-a)}$
$\mathsf{Gam}(\alpha,\lambda)$	$\left(\frac{\lambda}{\lambda-s}\right)^{\alpha}$
$\mathsf{N}(\mu,\sigma^2)$	$e^{s\mu+\sigma^2 s^2/2}$

41. **Moment property**: $\mathbb{E}X^n = M^{(n)}(0)$.

42. $M_{X+Y}(t) = M_X(t)\, M_Y(t)$, $\forall t$, if X, Y independent.

43. If $X_i \sim \mathsf{N}(\mu_i, \sigma_i^2)$, $i = 1, 2, \ldots, n$ (independent), then
$a + \sum_{i=1}^n b_i\, X_i \sim \mathsf{N}\left(a + \sum_{i=1}^n b_i\, \mu_i,\ \sum_{i=1}^n b_i^2\, \sigma_i^2\right)$.

44. **Conditional pmf/pdf**

$$f_{Y\,|\,X}(y\,|\,x) := \frac{f_{X,Y}(x,y)}{f_X(x)}, \quad y \in \mathbb{R}.$$

45. The corresponding **conditional expectation** (discrete case): $\mathbb{E}[Y\,|\,X = x] = \sum_y y\, f_{Y\,|\,X}(y\,|\,x)$.

46. **Linear transformation**: $f_{\boldsymbol{Z}}(\boldsymbol{z}) = \frac{f_{\boldsymbol{X}}(A^{-1}\boldsymbol{z})}{|A|}$.

47. **General transformation**: $\mathrm{d}s f_{\boldsymbol{Z}}(\boldsymbol{z}) = \frac{f_{\boldsymbol{X}}(\mathbf{x})}{|J_{\mathbf{x}}(g)|}$, with $\mathbf{x} = g^{-1}(\boldsymbol{z})$, where $|J_{\mathbf{x}}(g)|$ is the Jacobian of g evaluated at $\boldsymbol{x}$.

48. Pdf of the **multivariate normal** distribution:

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = \frac{1}{\sqrt{(2\pi)^n\, |\Sigma|}}\, e^{-\frac{1}{2}(\boldsymbol{z}-\boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{z}-\boldsymbol{\mu})} .$$

Σ is the covariance matrix, and $\boldsymbol{\mu}$ the mean vector.

49. If $\mathbf{X}$ is a column vector with independent $\mathsf{N}(0,1)$ components, and B is a matrix with $\Sigma = BB^T$ (such a B can always be found), then $\mathbf{Z} = \boldsymbol{\mu} + B\mathbf{X}$ has a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ.

50. **Weak Law of Large Numbers:**

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) = 0, \quad \forall \varepsilon .$$

51. **Strong Law of Large Numbers**:

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{S_n}{n} = \mu\right) = 1,$$

as $n \to \infty$.

52. **Central Limit Theorem**:

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leqslant x\right) = \Phi(x),$$

where Φ is the cdf of the standard normal distribution.

53. **Normal Approximation to Binomial:** If $X \sim \mathsf{Bin}(n,p)$, then, for large n, $\mathbb{P}(X \leqslant k) \approx \mathbb{P}(Y \leqslant k)$, where $Y \sim \mathsf{N}(np, np(1-p))$.

Other Mathematical Formulas

1. Factorial. $n! = n\,(n-1)\,(n-2)\cdots 1$. Gives the number of *permutations* (orderings) of $\{1, \ldots, n\}$.

2. Binomial coefficient. $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$. Gives the number *combinations* (no order) of k different numbers from $\{1, \ldots, n\}$.

3. Newton's binomial theorem: $(a+b)^n = \sum_{k=0}^n a^k\, b^{n-k}$.

4. Geometric sum: $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ $(a \neq 1)$.
If $|a| < 1$ then $1 + a + a^2 + \cdots = \frac{1}{1-a}$.

5. Logarithms:
   (a) $\log(x\,y) = \log x + \log y$.
   (b) $e^{\log x} = x$.

6. Exponential:
   (a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.
   (b) $e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$.
   (c) $e^{x+y} = e^x\, e^y$.

7. Differentiation:
   (a) $(f+g)' = f' + g'$,
   (b) $(fg)' = f'g + fg'$,
   (c) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.
   (d) $\frac{d}{dx} x^n = n\, x^{n-1}$.
   (e) $\frac{d}{dx} e^x = e^x$.
   (f) $\frac{d}{dx}\log(x) = \frac{1}{x}$.

8. Chain rule: $(f(g(x)))' = f'(g(x))\, g'(x)$.

9. Integration: $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$, where $F' = f$.

10. Integration by parts:
$\int_a^b f(x)\,G(x)\,dx = [F(x)\,G(x)]_a^b - \int_a^b F(x)\,g(x)\,dx$.
(Here $F' = f$ and $G' = f$.)

11. Jacobian: Let $\mathbf{x} = (x_1, \ldots, x_n)$ be an n-dimensional vector, and $g(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. The *matrix of Jacobi* is the matrix of partial derivatives: $(\partial g_i/\partial x_j)$. The corresponding determinant is called the *Jacobian*. In the neighbourhood of any fixed point, g behaves like a *linear transformation* specified by the matrix of Jacobi at that point.

12. Γ function: $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1}\, e^{-u}\,du$, $\alpha > 0$.
$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$, for $\alpha \in \mathbb{R}_+$. $\Gamma(n) = (n-1)!$ for $n = 1, 2, \ldots$. $\Gamma(1/2) = \sqrt{\pi}$.